SUBJECT: MATHEMATICS
TOPIC/MODULE: ORDINARY DIFFERENTIAL EQUATIONS

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REVIEW:

- ❖ Introduction to differential equations
- ❖ Basic concepts and definitions
- ❖ First order differential equations
- ❖ Second order differential equations
- ❖ Existence and Uniqueness Theorem
- ❖ Higher order differential equations
- ❖ Stability theory

INTRODUCTION.

Differential equations can describe nearly all systems undergoing change. They are ubiquitous is science and engineering as well as economics, social science, biology, business, healthcare, etc. Many mathematicians have studied the nature of these equations for hundreds of years and there are many well-developed solution techniques.

Often, systems described by differential equations are so complex, or the systems that they describe are so large, that a purely analytical solution to the equations is not tractable. It is mainly here so we can get some basic definitions and concepts out of the way.

In the sciences and engineering, mathematical models are developed to aid in the understanding of physical phenomena. These models often yield an equation that contains some derivatives of an unknown function. Such an equation is called a **differential equation.**

Isaac Newton (1643–1727),

Newton's laws were verified by experiment and observation for over 200 years, and they are excellent approximations at the scales and speeds of everyday life. Newton's laws of motion, together with his law of universal gravitation and the mathematical techniques of calculus, provided for the first time a unified quantitative explanation for a wide range of physical phenomena.

First law:
When viewed in an inertial reference frame, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a net force.

Second law:
In an inertial reference frame, the vector sum of the forces **F** on an object is equal to the mass m of that object multiplied by the acceleration vector **a** of the object: **F** = m**a**.

Third law:
To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

BASIC CONCEPTS DEFINITIONS

BASIC CONCEPTS:

There is one differential equation that everybody probably knows, that is Newton's Second Law of Motion. If an object of mass m is moving with acceleration a and being acted on with force F then Newton's Second Law tells us.

$$F = ma \qquad\qquad (1)$$

To see that this is in fact a differential equation we need to rewrite it a little. First, remember that we can rewrite the acceleration, a, in one of two ways.

$$a = \frac{dv}{dt} \qquad (or) \qquad a = \frac{d^2u}{d^2t} \qquad (2)$$

Where v is the velocity of the object and u is the position function of the object at any time t. We should also remember at this point that the force, F may also be a function of time, velocity,

So, with all these things in mind Newton's Second Law can now be written as a differential equation in terms of either the velocity, *v*, or the position, *u*, of the object as follows.

$$m\,\frac{dv}{dt} = F(t,v) \tag{3}$$

$$m\,\frac{d^2u}{d^2t} = F\left(t,v,\frac{du}{dt}\right) \tag{4}$$

So, here is our first differential equation.

Here are a few more examples of differential equations.

$$ay'' + by' + cy = g(t) \tag{5}$$

$$\sin(y) = \frac{d^2y}{dx^2} = (1-y)\frac{dy}{dx} + y^2e^{-5y} \tag{6}$$

DEFINITIONS:

DIFFERENTIAL EQUATIONS:

A differential equation is an equation involving an unknown function and its derivatives.

Example: The following are differential equations involving the unknown function y.

$$\frac{dy}{dx} = 5x + c \qquad\qquad (7)$$

ORDINARY DIFFERENTIAL EQUATIONS:

A differential equation is an *ordinary differential equation* if the unknown function depends on only one independent variable.

Examples :

$$\frac{dv(t)}{dt} - v(t) = e^t$$

$$\frac{d^2 x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

x(t):unknown function

t:independent variable

PARTIAL DIFFERENTIAL EQUATIONS:

      A differential equation is an *partial differential equation* if the unknown function depends on two or more independent variables.

Example:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \sin x \qquad (9)$$

DIFFERENTIAL EQUATIONS

Differential Equations

Ordinary Differential Equations

$$\frac{d^2 v}{dt^2} + 6tv = 1$$

unknown functions and

its derivative of one independent

variable

Partial Differential Equations

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

unknown functions and

its derivative of

two or more independent variable

ORDER

The *order* of a differential equation is the order of the highest derivative found in the Differential equation

$$\frac{d^2 y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

Second order

DEGREE:

The *degree* of a differential equation is power of highest order derivative in term in the differential equation. power of the highest order derivative term

Differential Equation	Degree
$\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + ay = 0$	1
$\dfrac{d^3y}{dx^3} + \left(\dfrac{dy}{dx}\right)^4 + 6y = 3$	1
$\left(\dfrac{d^2y}{dx^2}\right)^3 + \left(\dfrac{dy}{dx}\right)^5 + 3 = 0$	3

GENERAL SOLUTION:

Solutions obtained from integrating the differential equations are called general solutions. The general solution of a nth order ordinary differential equation contains n arbitrary constants resulting from integrating times.

PARTICULAR SOLUTION:

Particular solutions are the solutions obtained by assigning specific values to the arbitrary constants in the general solutions.

SINGULAR SOLUTION:

Solutions that can not be expressed by the general solutions are called singular solutions.

INITIAL CONDITION:

Constrains that are specified at the initial point, generally time point, are called initial conditions. Problems with specified initial conditions are called initial value problems.

BOUNDARY CONDITION:

Constrains that are specified at the boundary points, generally space points, are called boundary conditions. Problems with specified boundary conditions are called boundary value problems.

HOMOGENEOUS EQUATION:

In a differential equation if $r(x) = 0$ (that is, $r(x) = 0$ for all x considered; read "$r(x)$ is identically zero", then (*) reduces to

$$y'' + p(x)y' + q(x)y = r(x) \qquad \rightarrow \quad (*)$$

and is called **homogeneous equation**. If $r(x) \neq 0$, then (*) is called **nonhomogeneous equation**.

INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS

Initial-Value Problems

☐ The auxiliary conditions are at one point of the independent variable

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad \dot{x}(0) = 2.5$$

same

Boundary-Value Problems

☐ The auxiliary conditions are not at one point of the independent variable

☐ More difficult to solve than initial value problems

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad x(2) = 1.5$$

different

FIRST ORDER DIFFERENTIAL EQUATIONS

The most general first order differential equation can be written as,

$$\frac{dy}{dx} = f(t, y) \qquad\qquad (10)$$

there is no general formula for the solution to (10). What we will do instead is look at several special cases and see how to solve those. We will also look at some of the theory behind first order differential equations as well as some applications of first order differential equations.

SEPARABLE EQUATIONS:

The general solution to the first-order separable differential equation

$$A(x)\,dx + B(y)\,dy = 0 \qquad (11)$$

$$\int A(x)\,dx + \int B(y)\,dy = c \qquad (12)$$

where c represents an arbitrary constant.

The integrals obtained in Equation (12) may be, for all practical purposes, impossible to evaluate.

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is separable if the equation can be written in the form:

$$f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0$$

Solution :

1. Multiply the equation by integrating factor:

$$\frac{1}{f_2(x)g_1(y)}$$

2. The variable are separated :

$$\frac{f_1(x)}{f_2(x)}dx + \frac{g_2(y)}{g_1(y)}dy = 0$$

3. Integrating to find the solution:

$$\int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(y)}{g_1(y)}dy = C$$

EXAMPLE:

Find the general solutions of the following equations giving your answers in the form: $y=f(x)$

$$(1 + x^2)\frac{dy}{dx} = 2x$$

Solution:

$$(1 + x^2)\frac{dy}{dx} = 2x$$

$$dy = \frac{2x}{1 + x^2}dx$$

$$y = \log(1 + x^2) + C$$

HOMOGENEOUS EQUATION

$$y'' + P(x)y' + Q(x)y = 0 \qquad ------(*)$$

○ **Theorem:** y_1, y_2 solutions of Eq. (*)

$$\Rightarrow c_1 y_1(x) + c_2 y_2(x) \text{ solution of Eq. (*)}$$

c_1, c_2 real numbers

Proof:

$$y'' + P(x)y' + Q(x)y$$

$$= (c_1 y_1 + c_2 y_2)'' + P(x)(c_1 y_1 + c_2 y_2)' + Q(x)(c_1 y_1 + c_2 y_2)$$

$$= c_1 y_1'' + c_2 y_2'' + c_1 P(x) y_1' + c_2 P(x) y_2'$$

$$+ c_1 Q(x) y_1 + c_2 Q(x) y_2$$

$$= c_1[y_1'' + P(x)y_1' + Q(x)y_1] + c_2[y_2'' + P(x)y_2' + Q(x)y_2]$$

$$= 0 + 0 = 0$$

Example: $\quad y_1(x) = \cos x, \quad y_2(x) = \sin x$

are solutions of $\quad y'' + y = 0$

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1 \neq 0$$

$$\therefore \ y_1, \ y_2 : \text{linearly independent}$$

NONHOMOGENEOUS EQUATION:

$$y'' + P(x)y' + Q(x)y = R(x)$$

○ **Theorem :**

y_1, y_2 : linearly independent homogeneous

solutions of $\quad y'' + P(x)y' + Q(x)y = 0$

$y_p \qquad$: a nonhomogeneous solution of

$$y'' + P(x)y' + Q(x)y = R(x)$$

any solution φ has the form

$$\varphi = c_1 y_1 + c_2 y_2 + y_p$$

Proof: Given φ, y_p : solutions

$$(\varphi - y_p)'' + P(\varphi - y_p)' + Q(\varphi - y_p)$$

$$= \varphi'' + P\varphi' + Q\varphi - (y_p'' + Py_p' + Qy_p)$$

$$= R - R = 0$$

$\therefore$ $\varphi - y_p$ a homogenous solution of

$$y'' + P(x)y' + Q(x)y = 0$$

Q y_1, y_2 linearly independent homogenous solutions

$$\varphi - y_p = c_1 y_1 + c_2 y_2$$

$$\varphi = c_1 y_1 + c_2 y_2 + y_p$$

LINEAR DIFFERENTIAL EQUATIONS

The standard form of a linear differential equation of first order and first degree is

$$\frac{dy}{dx} + py = Q$$

where P and Q are the functions of x, or constants.

LINEARITY

➢ The important issue is how the unknown *y* appears in the equation. A linear equation involves the dependent variable (*y*) and its derivatives by themselves. There must be no "unusual" nonlinear functions of *y* or its derivatives.

➢ A linear equation must have constant coefficients, or coefficients which depend on the independent variable (*t*). If *y* or its derivatives appear in the coefficient the equation is non-linear.

EXAMPLES:

$$\frac{dy}{dt} + y = 0 \quad \text{is linear}$$

$$\frac{dx}{dt} + x^2 = 0 \quad \text{is non-linear}$$

$$\frac{dy}{dt} + t^2 = 0 \quad \text{is linear}$$

$$y\frac{dy}{dt} + t^2 = 0 \quad \text{is non-linear}$$

LINEAR DIFFERENTIAL EQUATIONS

Rule for solving $\dfrac{dy}{dx} + py = Q$

where P and Q are the functions of x, or constants.

Integrating factor (I.F) = $e^{\int pdx}$

The solutions is

$$y\,(I.F) = \int \{Q \times I.F\}dx + c$$

Example:

     Solve the differential equation $\dfrac{dy}{dx} + 2x = 6e^x$

Solution:

The given differential equation is $\dfrac{dy}{dx} + 2x = 6e^x$

It's linear equation of the form

Here P=2 and Q= $6e^x$

$$\dfrac{dy}{dx} + py = Q$$

$$I.F = e^{\int pdx} = e^{\int 2dx} = e^{2x}$$

$$y(I.F) = \int \{Q \times I.F\}dx + c$$

The solution is given by

$$\Rightarrow y(e^{2x}) = \int 6e^x \times e^{2x}dx + c$$

$$\Rightarrow ye^{2x} = \int 6e^{3x}dx + c$$

$$\Rightarrow ye^{2x} = 6\int e^{3x}dx + c$$

$$\Rightarrow ye^{2x} = 6 \times \frac{3x}{3}dx + c$$

$$\Rightarrow ye^{2x} = 2e^{3x}dx + c$$

$$\Rightarrow y = 2e^{3x} + c \times e^{-2x}$$

SECOND OREDR DIFFERENTIAL EQUATIONS

PRELIMINARY CONCEPTS:

◯ Second-order differential equation

$$F(x, y, y', y'') = 0$$

Example: $y'' - 3y' + 10y - 7x + 4 = 0$

$$y'' - 12x = 0$$

Solution: A function $\varphi(x)$ satisfies

$$F(x, \varphi(x), \varphi'(x), \varphi''(x)) = 0, \quad x \in I$$

$(I$: an interval$)$

SECOND ORDER LINEAR HOMOGENEOUS EQUATION:

A second order linear homogeneous equation has the form:

where a_2, a_1, a_0 are constants

To solve such an equation:

Let

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

$$D = \frac{d}{dx}$$

$$\frac{d^2 y}{dx^2} + c_1 \frac{dy}{dx} + c_2 y = 0$$

$$\Rightarrow (D^2 + K_1 D + K_2) y = 0$$

$$\Rightarrow (D - a)(D - b) y = 0$$

$$y = \begin{cases} C_1 e^{ax} + C_2 e^{bx} & \text{if } a \neq b \\ (Ax + B) e^{ax} & \text{if } a = b \end{cases}$$

LINEAR SECOND ORDER DE

➢ Homogeneous Equation:

$$a(t)x'' + b(t)x' + c(t)x = 0$$

➢ Non-Homogeneous Equation:

$$a(t)x'' + b(t)x' + c(t)x = f(t)$$

LINEAR SECOND-ORDER CONSTANT COEFFICENT DE

➢ Homogeneous Equation:

$$ax'' + bx' + cx = 0$$

➢ Non-Homogeneous Equation:

$$ax'' + bx' + cx = f(t)$$

Example:

Solve the differential equation: $\dfrac{d^2y}{dx^2} = \dfrac{1}{x} + 6.$

Solution: The given differential equation is

$$\dfrac{d^2y}{dx^2} = \dfrac{1}{x} + 6 \Rightarrow \dfrac{dy}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{x} + 6$$

$$\therefore \int \dfrac{dy}{dx}\left(\dfrac{dy}{dx}\right) dx = \int \dfrac{1}{x} dx + 6 \int dx$$

$$\Rightarrow \dfrac{dy}{dx} = \log_e |x| + 6x + C_1$$

where C_1 is constant of integration.

$$\therefore \int \left(\frac{dy}{dx}\right) dx = \int \log_e |x|.1 \, dx + 6\int x \, dx + C_1 \int dx$$

$$\Rightarrow y = x\log_e |x| - \int \frac{1}{x}.x \, dx + 6.\frac{x^2}{2} + C_1 x + C_2$$

where c_2 is constant of integration

$$\Rightarrow y = x\log_e |x| - x + 3x^2 + C_1 x + C_2$$

$$\Rightarrow y = x\log_e |x| + 3x^2 + x(C_1 - 1) + C_2$$

This is required solution.

EXISTENCE AND UNIQUENESS

The general first-order ODE is

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

➢ Does there have to be a solution?

➢ If so, could there be more than one solution?

➢ Does the solution is unique?

Existence and uniqueness theorem for first order ode's

Let *R* be a rectangle and let *f(x, y)* be continuous throughout *R* and satisfy the Lipschitz Condition with respect to *y* throughout *R*. Let (x_0, y_0) be an interior point of *R*. Then there exists an interval containing x_0 on which there exists a unique function *y(x)* satisfying

$$y_0 = f(x, y) \qquad y(x_0) = y_0$$

HOMOGENEOUS EQUATIONS:

The general form of n-th order homogeneous linear equations

$$a_n y^n + a_{n-1} y^{n-1} + \ldots + a_1 y' + a_0 y = 0$$

where $a_n, \ldots, a_1, a_0$ are constants with $a_n \neq 0$

Solution Method:

➢ Find the roots of the characteristic polynomial:
$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_n \lambda^n + a_1 \lambda + a_1 = 0$$

➢ Each root λ produces a particular exponential solution $e^{\lambda t}$ of the differential equation.

➢ A repeated root λ of multiplicity k produces k linearly independent

   solutions $e^{\lambda t}, te^{\lambda t}, \ldots, t^{k-1} e^{\lambda t}$

Example 1:

Find general solutions of $y''' + 4y'' - 7y' - 10y = 0$.

Solution:

Solve the characteristic polynomial:

$$\lambda^3 + 4\lambda^2 - \lambda - 10 = 0$$

The roots are $\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = -5$ Each root gives a particular exponential solution of the differential equation. Combined, the general solutions are

$$y = C_1 e^{-t} + C_2 e^{2t} + C_3 e^{-5t}$$

where C_1, C_2, C_3 are arbitrary constants.

.

Example 2:

Salve $y''' + 4y'' - 7y' - 10y = 0$ with initial conditions $y(0) = -3$, $y'(0) = 12$, $y''(0) = -36$.

Solution:

We use the initial conditions to determine the values of the constants in the general C_1, C_2, C_3

solution formula. The initial conditions are

$y(0) = -3$, $y'(0) = 12$, $y''(0) = -36$ yield

$$C_1 + C_2 + C_3 = -3$$

$$-C_1 + C_2 + C_3 = 12$$

$$C_1 + 4C_2 + 25C_3 = -36$$

Solve this linear system for C_1, C_2, C_3

$$C_1 = -\frac{5}{2}$$

$$C_2 = 1$$

$$C_3 = -\frac{3}{2}$$

Thus, the solution to the initial value problem is

$$y = -\frac{5}{2}e^{-t} + e^{2t} - \frac{3}{2}e^{-5t}$$

NON-HOMOGENEOUS EQUATIONS:

The general form of n-th order Non- homogeneous linear equations

$$a_n y^n + a_{n-1} y^{n-1} + \ldots + a_1 y' + a_0 y = f(t)$$

where $a_n, \ldots, a_1, a_0$ are constants with $a_n \neq 0$ and $f(t)$ is a given function.

Solution method:

The general solutions of the nonhomogeneous equation are of the following structure:

$$y = y_c + y_p$$

where y_c (the so-called "complementary" solutions) are solutions of the corresponding

Example: Solve $y''' + 4y'' - 7y' - 10y = 100t^2 + 64e^{3t}$

Solution:

First, solve the corresponding homogeneous equation

$$y_c''' + 4y_c'' - 7y_c' - 10y_c = 0$$

The complementary solutions are

$$y_c = C_1 e^{-t} + C_2 e^{2t} + C_3 e^{-5t}$$

where C_1, C_2, C_3 are arbitrary constant

Next, set up a trial function by copying the structure of $f(t)$

Substitute this into the nonhomogeneous equation and simplify

$$y_0 = a_0 + a_1 t + a_2 t^2 + be^{3t}$$

$$(-10a_0 - 7a_1 + 8a_2) + (-10a_1 - 14a_2)t \quad -10a_2t^2 + 32be^{3t}$$
$$= 100t^2 + 64e^{3t}$$

Compare the coefficients of the two sides

$$-10a_0 - 7a_1 + 8a_2 = 0,$$
$$-10a_1 - 14a_2 = 0,$$
$$-10a_2 = 100,$$
$$32b = -64.$$

Solving this linear system we obtain

$$a_0 = \frac{-89}{5}, a_1 = 14, a_2 = -10, b = -2$$

and thus $\qquad y_p = \dfrac{-89}{5} + 14t - 10t^2 - 2e^{3t}$

The general solutions to the nonhomogeneous equation are

$$y = y_c + y_p$$

$$y = \dfrac{-89}{5} + 14t - 10t^2 - 2e^{3t} + C_1 e^{-t} + C_2 e^{2t} + C_3 e^{-5t}$$

Where C_1, C_2, C_3 are arbitrary constant

This is required solution

LINEAR ODE

❖ **LINEARITY:**

An equation Involves no product nor nonlinear functions of *y* and its derivatives

❖ n^{th} **ORDER LINEAR ODE:**

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \Lambda + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

❖ **ODE (IVP)**

First order ODE (<u>canonical</u> form)

$$\frac{dy}{dx} = f(x, y), \; y(x = 0) = y_0$$

Every n^{th} order ODE can be converted to *n* first order ODEs in the following method:

CONTINUE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \Lambda + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

$$y_1(x) \equiv y(x)$$

$$y_2(x) \equiv \frac{dy_1}{dx} = \frac{dy}{dx}$$

$$y_3(x) \equiv \frac{dy_2}{dx} = \frac{d^2 y}{dx^2}$$

$$M$$

$$y_n(x) \equiv \frac{dy_{n-1}}{dx} = \frac{d^{n-1} y}{dx^{n-1}}$$

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = y_3$$

$$M$$

$$\frac{dy_{n-1}}{dx} = y_n$$

$$a_n(x)\frac{dy_n}{dx} + a_{n-1}(x)y_n + \Lambda + a_1(x)y_2 + a_0(x)y_1 = b(x)$$

Example:

A population grows at the rate of 5% per year. How long does it take for the population to double? Use differential equation for it.

Solution: Let the initial population be P_0 and let the population after t years be P, then

$$\frac{dP}{dt} = \left(\frac{5}{100}\right)P \Rightarrow \frac{dP}{dt} = \frac{P}{20} \Rightarrow \frac{dP}{P} = \frac{1}{20} \ dt$$

$$\Rightarrow \int \frac{dP}{P} = \frac{1}{20} \int \ dt \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \log_e P = \frac{1}{20} t + C$$

At $t = 0$, $P = P_0$ $\qquad \therefore \log_e P_0 = \dfrac{1 \times 0}{20} + C \Rightarrow C = \log_e P_0$

$\therefore \log_e P = \dfrac{1}{20} t + \log_e P_0 \Rightarrow t = 20 \log_e \left(\dfrac{P}{P_0} \right)$

When $P = 2P_0$, then

$\Rightarrow t = 20 \log_e \left(\dfrac{2P_0}{P_0} \right) = \dfrac{1}{20} \log_e 2$ years

Hence, the population is doubled in

STABILITY THEORY

INTRODUCTION

❖ The concept of stability, which characterizes the long-term behavior of systems evolving in time, is introduced. The theory of Lyapunov which allows to rigorously deduce asymptotic stability is of particular interest in this context.

❖ The term "stable" informally means *resistant to change*. For technical use the term has to be defined more precisely in term of the mathematical model, but the same connotation applies.

❖ As we know, the stability is the most fundamental problem in the design of automatic control systems, since only a stable system can keep working properly under disturbances

❖ When one designs a control system, one first needs to consider some type of stability for the system and then investigate other problems. Among various stability theories, the Lyapunov stability is still the most important one

❖ The main difficulty in analyzing Lyapunov stability is how to determine a Lyapunov function for a given system. There does not exist general rules for constructing Lyapunov functions, but are merely based on a researcher or designer's experience and some particular techniques.

❖ Moreover, it should be pointed out that the Lyapunov stability theory is mainly applicable for local stability, while many practical problems need to consider globally asymptotic stability or even globally exponential stability.

❖ It is used in fluid dynamics teaching and in engineering as a simplified model for boundary layer behavior, shock wave formation, and mass transport. It has been studied and applied for many decades.

Asymptotically stable:

An equilibrium state x_e *of an autonomous system is **asymptotically stable** if* $\delta_a \longrightarrow 0$ *if*

➢ *It is stable and exist*

$$\left\| x_0 - x_e \right\| \pi \; \delta_a \Longrightarrow \left\| x(t) - x_e \right\| \longrightarrow 0, as \quad t \longrightarrow \infty$$

Asymptotically stable in the large

*(**globally asymptotically stable**)*

➢ The system is asymptotically stable for all the initial states

.

➢ The system has only one equilibrium state.

Theorem:

$\dot{x} = f(x,t)$ with equilibrum point $x_e = 0$, if $\exists\, a\ scalar\ function\ v(x,t)\ satisfies$

(1) $v(x,t)\ has\ continuous\ first\ order\ derivative$

(2) $v(x,t)\ is\ positive\ definite$

(3) $\dot{v}(x,t)\ is\ negative\ semidefinite$

(4) $\dot{v}(x,t)\ is\ not\ identically\ zero\ when\ x \neq 0$

$then, the\ origion\ is\ uniformly\ asymptotically\ stable.$

Example:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 \end{cases}, \ x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$Let \ \ v(x) = x_1^2 + x_2^2 > 0$$

$$\dot{v}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2x_2^2 \le 0$$

$$when \ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}, \ \ \dot{v}(x) = 0$$

$$and \ \ \dot{x}_2 = -2x_1 - x_2 = * \neq 0$$

$$x_2 \ is \ not \ identically \ zero \Rightarrow \dot{v}(x) = -2x_2^2 \ is \ not \ identically \ zero$$

$$\Rightarrow the \ origion \ is \ asymptotically \ stable.$$

Equilibrium Points

- Equilibrium points represent stationary conditions for the dynamics.

- The equilibrium points (equilibria) of the dynamic system

$$\dot{x} = f(x)$$

are the points x_e such that

$$f(x_e) = 0.$$

Simple Pendulum Example

- **Consider the dynamics of a simple pendulum**

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{1}{ml^2}T_a$$

where θ is the angular displacement and T_a is the applied torque at the pivot point.

- **Equilibrium points are given by**

$$\sin\theta_{eq} = 0 \quad \text{or} \quad \theta_{eq} = \pm n\pi, \; n = 0, 1, 2, \ldots$$

- **Physically, pendulum straight up and pendulum straight down are all the equilibrium points.**

Stability of Equilibrium Points

- An equilibrium point is <u>stable</u> if all initial conditions that start near the equilibrium point, stay near it.

- An equilibrium point is <u>unstable</u> if some initial conditions diverge from the equilibrium point.

- An equilibrium point is <u>asymptotically stable</u> if all initial conditions converge to the equilibrium point.

- Nonlinear systems, in general, may have multiple equilibrium points; for example, take the simple pendulum.

Stability of Equilibrium Points

An equilibrium point is:

Asymptotically stable if all nearby initial conditions converge to the equilibrium point

- Equilibrium point is an *attractor* or *sink*

$$\lim_{t \to \infty} x(t) = x_e \ \forall \ \|x(0) - x_e\| < \varepsilon$$

Unstable if some initial conditions diverge from the equilibrium point

- Equilibrium point is a *source* (or *saddle*)

$$\lim_{t \to \infty} \|x(t)\| = \infty \text{ for some } x(0)$$

Stable if initial conditions that start near the equilibrium point, stay near

- Equilibrium point is a *center*
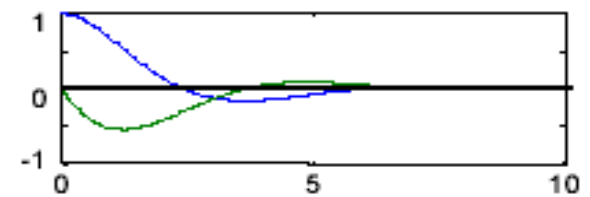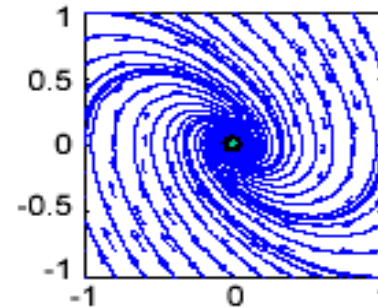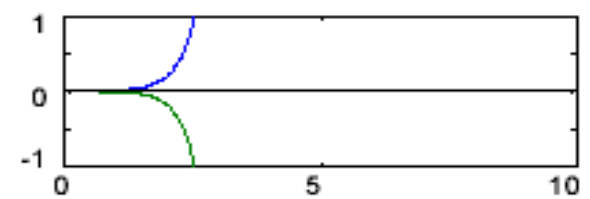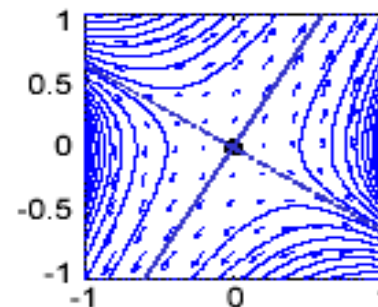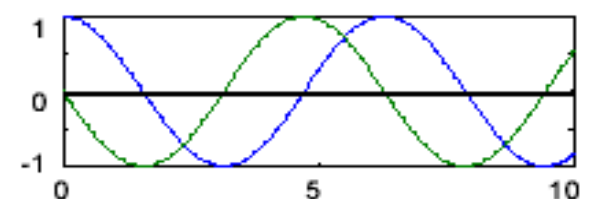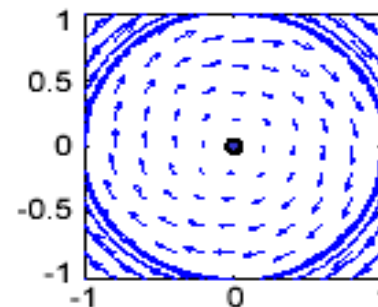
$$\|x(t) - x_e\| < \varepsilon \ \forall t, \|x(0) - x_e\| < \delta_\varepsilon$$

A ball bearing, with dissipative friction, in a gravity field:

Asymptotically Stable

Stable in the Sense of Lyapunov

Unstable

Lyapunov A. M. (1857-1918)

Alexander Lyapunov was born on 6 June 1857 in Yaroslavl, Russia in the family of the famous astronomer M.V. Lyapunov, who played a great role in the education of Alexander and Sergey.

Alexander Lyapunov was a school friend of Markov and later a student of Chebyshev at Physics & Mathematics department of Petersburg University which he entered in 1976. He attended the chemistry lectures of D.Mendeleev.

In 1985 he brilliantly defends his MSc diploma "On the equilibrium shape of rotating liquids", which attracted the attention of physicists, mathematicians and astronomers of the world.

The same year he starts to work in Kharkov University at the Department of Mechanics. He gives lectures on Theoretical Mechanics, ODE, Probability.

In 1892 defends PhD. In 1902 was elected to Science Academy.

After wife's death 31.10.1918 committed suicide and died 3.11.1918.

Lyapunov stability

- Let $x = 0$ be an equilibrium for $\dot{x} = f(x)$ (i.e., $f(0) = 0$).

- Let $D \subset \mathbb{R}^n$ be a region containing the origin and $V : D \to \mathbb{R}_+$ a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D\backslash\{0\} \qquad \text{i.e. } V \text{ is positive definite}$$

$$\dot{V}(x) \leq 0 \text{ in } D \qquad \text{i.e. } \dot{V} \text{ is negative semi-definite}$$
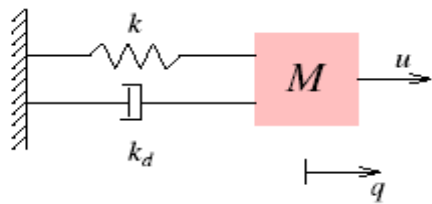
Then $x = 0$ is stable, and V is termed a *Lyapunov Function*.

- If

$$\dot{V}(x) < 0 \text{ in } D\backslash\{0\}$$

then $x = 0$ is asymptotically stable.

Mass–spring system—Lyapunov stability

$$\dot{q} = p$$

$$\dot{p} = -q - p$$

- Energy of the system: $E = \frac{1}{2}p^2 + \frac{1}{2}q^2$, and $\dot{E} = -p^2 \leq 0$.
  $\Rightarrow (q, p) = 0$ is stable.

- Let $V = 1.5q^2 + qp + p^2$. Is V positive definite?
  Yes, because $V = (\frac{1}{2}q + p)^2 + \frac{5}{4}q^2 > 0$, for all $(p, q) \neq 0$.
  And $\dot{V} = -(q^2 + p^2) < 0$, for all $(p, q) \neq 0$.
  $\Rightarrow (q, p) = 0$ is asymptotically stable (as expected).

Lyapunov stability—Barrier functions

- Given a Lyapunov function, the direct method is an easy way of proving stability of equilibria, as Lyapunov's theorem can be used without solving the differential equations.

- However, there are no systematic ways to construct Lyapunov functions, apart from special cases (energy functions for mechanical systems, etc). Moreover, testing non-negativity of a function is *NP-hard*.

- We call the surface $V(x) = c$ for some $c > 0$ a level surface.

- Condition $\dot{V} \leq 0$ implies that when a trajectory crosses the level surface $V(x) = c$ it moves inside the set $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ and never comes out again.