

physics751: Group Theory (for Physicists)

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Chapter 1

Preliminaries

1.1 About These Notes

These are the full lecture notes, i.e. including up to Chapter 6, which deals with representations of Lie algebras. Most likely there are still errors – please report them to me so I can fix them in later versions!

When reading the notes, you should also do the exercises which test your understanding and flesh out the concepts with explicit examples and calculations. Doing the exercises is what makes you learn this (and any other) topic!

1.2 Time and Place

This is a three-hour course. Lectures will be on Wednesday from 14 to 16 and on Friday from 9 to 10. Since lecture period is from April 12 to July 23, with a break from May 25 to 29, i.e. 14 weeks, and Dies Academicus on Wednesday May 19, there will be 13 two-hour and 14 one-hour lectures.

The problem sheets, lecture notes (chapter by chapter, delayed with respect to the lecture) and further information will be posted on <http://www.th.physik.uni-bonn.de/nilles/people/luedeling/grouptheory/>

1.3 Tutorials

There are three tutorial groups:

- Monday 10–12, Raum 5 AVZ, Thomas Wotschke
- Tuesday 10–12, Raum 5 AVZ, Daniel Lopes
- Wednesday 10–12, Raum 118 AVZ (Hörsaal), Marco Rauch.

Problems will be handed out in week n , solved during the week, collected and discussed in week $n + 1$, corrected during the next week and returned in week $n + 2$. Problems can be solved in groups of two students.

1.4 Exam and Requirements

To be allowed in the exam, a student needs to have obtained 50 % of the problem sheet points AND presented two solution on the blackboard.

Exam will be written the on the first Tuesday after the term, July 27. If required, a resit will be offered at the end of the term break, i.e. late September or early October.

1.5 Literature

There are many books on this subject, ranging from formal to applied. Here I give a selection. Furthermore, many lecture notes are available on the web. In particular, I have partially followed the lecture notes of Michael Ratz (TU Munich), which are unfortunately not freely available on the web.

- H. Georgi, “Lie Algebras In Particle Physics. From Isospin To Unified Theories,” *Westview Press (Front. Phys. 54)* (1982) 1.
A classic, very accessible. A second edition has come out in 1999, containing also a nice chapter on discrete groups.
- M. Hamermesh, “Group Theory and Its Application to Physical Problems,” *Addison–Wesley Publishing (1962)*
A classical reference, in particular for discrete groups and applications in quantum mechanics.
- H. Weyl, “Quantum mechanics and group theory,” *Z. Phys. 46* (1927) 1.
One of the original foundations of the use of symmetry in quantum mechanics
- R. N. Cahn, “Semisimple Lie Algebras And Their Representations,” *Menlo Park, USA: Benjamin/Cummings (1984) 158 P. (Frontiers In Physics, 59)* (Available online at <http://phyweb.lbl.gov/~rncahn/www/liealgebras/book.html>)
Short book, available for free
- H. F. Jones, “Groups, representations and physics,” *Bristol, UK: Hilger (1990) 287 p*
Also discusses finite groups and quantum mechanics, mathematically simple
- R. Gilmore, “Lie Groups, Lie Algebras, and Some of Their Applications,” *New York, USA: Wiley Interscience (1974)*
Covers mainly mathematical aspects of Lie groups, supplies some proofs omitted in the lecture
- W. Fulton and R. Harris, “Representation Theory: A First Course”, *Springer Graduate Text in Mathematics 1991*
Modern Treatment, mathematical but very accessible

- J. Fuchs and C. Schweigert, “Symmetries, Lie Algebras And Representations: A Graduate Course For Physicists,” *Cambridge, UK: Univ. Pr. (1997) 438 p*
Rather formal and advanced treatment, for the mathematically interested
- R. Slansky, “Group Theory For Unified Model Building,” *Phys. Rept.* **79**, 1 (1981).
Invaluable reference: Contains a large appendix with loads of tables of representations and branching rules.
- M. Nakahara, “Geometry, topology and physics,” *Bristol, UK: Institute of Physics Publishing (2003) 596 p. (Graduate student series in physics)*
Very useful book, covers in particular the differential geometry aspects of Lie groups. Generally recommended for particle theorists.

Chapter 2

Groups: General Properties

2.1 Motivation: Why Group Theory?

Why are there lectures called “Group Theory for Physicists”? In the end, this is a mathematical subject, so why don’t students interested in the topic attend a mathematics lecture? After all, there are very few lectures like “Number Theory for Physicists”. This is captured in a statement made by James Jeans in 1910 while discussing a syllabus¹: “We may as well cut out the group theory. That is a subject that will never be of any use in physics.”

Only few decades later, however, Heisenberg said² “We will have to abandon the philosophy of Democritus and the concept of elementary particles. We should accept instead the concept of elementary symmetries.” This explains why group theory is important in almost any area of theoretical physics: It is the mathematics underlying the concept of symmetry. By symmetry, here we roughly mean “a transformation which leaves the physical situation unchanged”, and these naturally form groups (they can be composed and undone). This implies that physical quantities appear in “multiplets”, i.e. sets which transform among themselves. This is intuitively clear in Newtonian mechanics: You cannot sensibly add a scalar and the x -component of a vector, even if they have the same units (such as e.g. energy and torque), because they behave differently under a rotation of the frame of reference (the scalar is constant, the vector component mixes with the y and z components). Hence, the sum will depend on the observer in a weird way. But there are more things than scalars and vectors. In quantum mechanics, this becomes more important.

As an example, consider the hydrogen atom. The Hamiltonian is invariant under spatial rotations, i.e. if R is an operator corresponding to such a rotation, it commutes with the Hamiltonian, $[H_0, R] = 0$. Since the R ’s form a group, called $SO(3)$, this immediately tells us that the eigenstates of H_0 must come in representations of $SO(3)$. Group theory tells us that these representations are labelled by two numbers (l, m) , which we interpret as angular momentum and magnetic quantum number. Furthermore,

¹Quoted in D MacHale, *Comic Sections* (Dublin 1993)

²Quoted in E Maor, *To infinity and beyond* (Princeton 1991)

group theory tells us something about the allowed transitions: Since the decay of an excited state is mediated by a perturbed Hamiltonian $H = H_0 + H_1$ containing the electromagnetic field, the transition rate T between two states $|nlm\rangle$ and $|n'l'm'\rangle$ is proportional to

$$T \sim \langle nlm | H_1 | n'l'm' \rangle . \quad (2.1)$$

Again group theory gives us relations between different such operators in the form of the Wigner–Eckart theorem. In particular, for H_1 corresponding to dipole radiation, this gives the familiar selection rules: $T = 0$ unless $l = l' \pm 1$ and $m = m'$ or $m = m' \pm 1$. The power of symmetry is such that we do not even need to know anything about H beyond its behaviour under rotations to deduce this. On the other hand, the symmetry does not tell us the transition rate if the selection rules are satisfied.

This is a prototypical application of group theory: By identifying the symmetries, one can find good quantum numbers to describe a physical state. Furthermore, symmetry arguments tell us that many matrix elements vanish, or are given by linear combinations of others. However, group theory does not necessarily determine the actual value allowed matrix elements.

The outline of the course is as follows (unfortunately, I had to drop the Lorentz group for lack of time):

1. Preliminaries: Done
2. General properties of groups: I will define a group and various basic concepts we need later on.
3. Representations: Groups as such are just elements that can be multiplied. In physics, they appear as operators acting on something, and this goes under the name of representations. I will introduce the concept and a few basic properties.
4. Discrete groups: In this section I will discuss a class of groups which is crucial e.g. in condensed matter physics, and their representations.
5. Compact Lie groups and Lie algebras: Lie groups are the most important groups for particle physics. We will discuss some aspects of Lie groups, in particular the connection to Lie algebras, which are something like a “local version” of the groups. It turns out that the algebra captures most properties of the underlying group, up to some aspects of the topology. We will see how to go from the group to the algebra and back, and consider the matrix groups in more detail.
6. Representations of Lie algebras: The representation theory of Lie groups can be reduced to the representations of Lie algebras. We will discuss the classification of Lie algebras and their representations in general, and go into more detail for the classical groups, in particular $SU(n)$.

2.2 Definition, Examples

Definition 1. A group is a set of elements $G = \{g_i\}$ together with a map (the “group product”)

$$\begin{aligned} G \times G &\longrightarrow G \\ (g_1, g_2) &\longmapsto g_1 g_2 \end{aligned}$$

satisfying the following properties:

1. *Associativity:* $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ for all $g_i \in G$.
2. *Identity:* There is an element e (the “identity” or “unit element”) which satisfies $eg = g$ for all $g \in G$.
3. *Inverse:* For every $g \in G$, there is an “inverse element” g^{-1} , such that $g^{-1}g = e$.

Notes:

- The product is sometimes written as $g_1 \cdot g_2$ or $g_1 \circ g_2$. In concrete groups, it can be realised e.g. by addition or multiplication of numbers or matrices or by composition of maps. The specification of the product is part of the definition of the group, so in principle one should write the group as (G, \circ) if the product is denoted by \circ . However, we always omit the product symbol and assume it is clear from the context what product is being used.
- In general, $g_1 g_2 \neq g_2 g_1$!
- If $g_1 g_2 = g_2 g_1$ for all $g_1, g_2 \in G$, the group is called commutative or Abelian. In that case, the “product” is often called addition and written accordingly, i.e. $g_1 g_2 \equiv g_1 + g_2$, inverses are $g^{-1} = -g$ and the identity is denoted by zero, $g + 0 = g$.
- We have not specified the topology of G . There are two basic distinctions: discrete/continuous and (roughly speaking) finite volume/infinite volume.
 - For discrete groups, it makes sense to count the number of group elements. This is called the *order of the group*, $|G|$. Clearly, this can be finite or infinite. For finite groups, the product can be specified in a multiplication table. Note that the order of a group should not be confused with the *order of a group element*, $\text{ord } g$, which is the smallest positive integer p such that $g^p = e$.
 - Continuous groups are those that have a notion of distance, i.e. it makes sense to say two group elements are “arbitrarily close”. In practice, this means that the elements can be parameterised by a set of real³ parameters, $g(x_1, \dots, x_d)$, and $g(x + \epsilon)$ is very close to $g(x)$. Then clearly, continuous groups have infinitely many elements, so the order of such a group is not particularly

³In principle, rational numbers would also be allowed, but that is of no importance in physics.

interesting. There are two related concepts: dimensions and compactness, which we will properly define later on but which should be intuitively clear: The dimension is the number of parameters you need to describe the group, and compactness refers to whether the range of parameters is compact (for a reasonably chosen parameterisation). In this lecture, we will only consider finite-dimensional groups. Furthermore, we will restrict to groups which are smooth in a way we will later define. Still, these can be compact or non-compact. Furthermore, they need not be connected or simply connected.

In the definition, we have required only a “left identity” and a “left inverse”. Actually, one can show that they are also right identity and inverses, and they are unique:

- i. To show that $gg^{-1} = e$, denote the inverse of g^{-1} by $(g^{-1})^{-1}$. The expression

$$(g^{-1})^{-1} g^{-1} gg^{-1}$$

can be grouped in two ways,

$$\begin{aligned} \left((g^{-1})^{-1} g^{-1} \right) gg^{-1} &= e gg^{-1} = gg^{-1} \\ &= (g^{-1})^{-1} (g^{-1} g) g^{-1} = (g^{-1})^{-1} e g^{-1} = (g^{-1})^{-1} g^{-1} = e. \end{aligned}$$

Hence, $gg^{-1} = e$, or in other words, $(g^{-1})^{-1} = g$.

- ii. The left identity is also a right identity:

$$ge = g(g^{-1}g) = (gg^{-1})g = eg = g.$$

- iii. The identity is unique: Assume e and f are identity elements, i.e. $ge = eg = gf = fg = g$. Then

$$e = ef = f.$$

- iv. The inverse is unique: Assume h is another inverse of g , i.e. $hg = e$. Then we have

$$h = he = hgg^{-1} = eg^{-1} = g^{-1}.$$

- v. Furthermore, we have the “rearrangement theorem”: Let g be any group element. Then

$$gG := \{gg_i\} = G.$$

The proof is left to the reader.

2.2.1 Examples

1. Finite groups:

- (a) The n -th roots of unity form an Abelian group of order n under multiplication: Let $\theta = \exp\{2\pi i/n\}$. Then $G = \{\theta^p, p = 1, \dots, n\}$. The identity is $1 = \theta^n$, the inverse of θ^p is $\theta^{-p} \equiv \theta^{n-p}$. In such a group, θ is called generating element. Note that any element θ^q , where q and n are relatively prime, is again a generating element.
- (b) The integers under addition modulo n form a group, which is actually isomorphic to the previous one. This shows that identical group structures can be realised in different ways.
A group which is generated by a single element θ which satisfies $\theta^n = e$ is called the *cyclic group* of order n , denoted \mathbb{Z}_n .
- (c) In some sense the most general finite group is the *symmetric group* S_n , consisting of the permutation of n things: Cayley's theorem states that every finite group of order N is a subgroup of S_N . This group is not Abelian.

2. Infinite discrete groups:

- (a) $(\mathbb{Z}, +)$: Integer numbers under addition. This is the infinite analogue of the cyclic group, since it is generated by the single element 1 by repeated addition, and by including the inverse. For an element of finite order p , the inverse is automatically included as θ^{p-1} , but if $p = \infty$, it has to be included explicitly. (The integers do not form a group under multiplication, even after cutting out zero, due to the lack of inverses.)
- (b) The *modular group* $SL(2, \mathbb{Z})$, the group of integer-valued 2×2 matrices with unit determinant, under matrix multiplication. This group is important in string theory.

3. Continuous compact groups:

- (a) The *orthogonal group* $O(n)$, the group of real $n \times n$ matrices O satisfying $O^T O = \mathbb{1}$. This corresponds to rotations and reflections in n -dimensional Euclidean space \mathbb{R}^n .
- (b) The *unitary group* $U(n)$, the group of complex $n \times n$ matrices U satisfying $U^\dagger U = \mathbb{1}$. This corresponds to (holomorphic) rotations and reflections in \mathbb{C}^n .
- (c) Orthogonal matrices have determinant ± 1 , while unitary matrices have determinant $e^{i\phi}$, i.e. of modulus one. In both cases, there is a "special" version of the groups, $SO(n)$ and $SU(n)$, respectively, with the extra restriction that the matrices must have determinant equal to one.

The orthogonal groups are actually not connected: The determinant is a continuous function of the group elements, and since it can only take the values ± 1 , $O(n)$ must have (at least) two connected components, one of which is $SO(n)$, while the matrices with determinant -1 do not form a group.

4. Continuous non-compact groups:

- (a) The real (rational, complex) numbers under addition. If zero is excluded, they also form a group under multiplication.
- (b) The positive real (rational) numbers under multiplication
- (c) The *general linear group* $GL(n, K)$ of $n \times n$ matrices with entries from the field K having nonzero determinant. This again has as a subgroup the *special linear group*, which consists of those matrices with determinant one. We will only consider the cases $K = \mathbb{R}$ or $K = \mathbb{C}$.
- (d) The *Euclidean group* $E(n)$ (or $ISO(n)$), the symmetry group of (affine) n -dimensional Euclidean space. It consists of all transformations $\vec{x} \rightarrow O\vec{x} + \vec{b}$ with an orthogonal matrix O and a constant shift \vec{b} .
- (e) The orthogonal and unitary groups can be defined as those linear transformations of a (real or complex) n -dimensional vector space preserving a positive definite scalar product (the usual Euclidean or Hermitean scalar product). If this scalar product is changed such that it is no longer positive definite, the resulting symmetry group will be non-compact, an example being the Lorentz group.
- (f) An example important for quantum mechanics is the automorphism group $\text{Aut}(V)$ of any vector space V . For finite-dimensional vector spaces, this coincides with $GL(n, K)$.

2.3 Subgroups and Cosets

Definition 2. A subgroup H of G is a subset of elements which forms a group under the group product of G .

- Every subgroup must contain the identity.
- A subset $H \subset G$ is a subgroup if and only if for all $h_1, h_2 \in H$, $h_1 h_2^{-1} \in H$.
- $\{e\}$ and G itself are trivially subgroups.
- If G is Abelian, so is H . The converse is not true.
- Every element $g \in G$ generates an Abelian subgroup $\langle g \rangle$. For elements of finite order, we simply have $\langle g \rangle = \{g^p\}$, $p = 1, \dots, \text{ord } g$. For $\text{ord } g = \infty$, on the other hand, we would miss the inverses and the identity, so we have to explicitly include e and g^{-1} . Writing $e = g^0$ and $g^{-p} = (g^{-1})^p$, we again have $\langle g \rangle = \{g^p\}$, but this time $p \in \mathbb{Z}$. In both cases, the order of $\langle g \rangle$ is the order of g (i.e. $|\langle g \rangle| = \text{ord } g$).

Examples:

- The cyclic group \mathbb{Z}_6 has subgroups of order two and three, generated by θ^3 and θ^2 , respectively.
- $O(3)$ has a discrete subgroup given by $\{\mathbb{1}, -\mathbb{1}\}$. Furthermore it has a continuous family of subgroups given by those rotations that leave a given vector invariant.
- $U(2)$ has non-equivalent one-dimensional subgroups,

$$U(1)_A = \{e^{i\phi}\mathbb{1}\} \quad \text{and} \quad U(1)_B = \left\{ \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \right\}$$

(Non-equivalent means that they are not related by a similarity transformation.)

- For every group G , the *center* is the set of those elements which commute with all elements of G ,

$$Z(G) = \{h \in G | hg = gh \text{ for all } g \in G\} .$$

Clearly, $Z(G)$ is an Abelian subgroup.

Definition 3. Let $H = \{h_i\}$ be a subgroup of G and $g \in G$ any group element. The set

$$gH = \{gh_i\}$$

is called a *left coset* of H . Similarly, Hg is a *right coset*.

It is easy to show that:

- A coset gH is a subgroup if and only if $g \in H$, in which case $gH = H$.
- Two cosets g_1H and g_2H are either equal or disjoint.
- All cosets have the same number of elements.

2.4 Conjugates, Normal Subgroups

Definition 4. Two group elements g_1 and g_2 are called *conjugate*, written as $g_1 \sim g_2$, if there exists an element g such that

$$g_1 = gg_2g^{-1} .$$

Conjugacy is an equivalence relation, i.e. it is

- reflexive: i.e. $g \sim g$,
- symmetric: from $g_1 \sim g_2$ it follows that $g_2 \sim g_1$, and

iii. transitive: if $g_1 \sim g_2$ and $g_2 \sim g_3$, then $g_1 \sim g_3$.

Since conjugacy is an equivalence relation, we can form equivalence classes, called *conjugacy classes*: Given any group element g , its conjugacy class is the set of all conjugate elements:

$$[g] = \{h \in G \mid g \sim h\}$$

g is called a *representative* of the class.

- If g' is another element of $[g]$, it can also be used as the representative, i.e. $[g] = [g']$ if and only if $g \sim g'$.
- For Abelian groups, every element is its own conjugacy class.
- The identity element is always its own class.
- If g is of order p , every element of $[g]$ is also of order p .

Definition 5. A subgroup H is called a *normal subgroup* (or *invariant subgroup*) if it is *self-conjugate*, i.e. if

$$gHg^{-1} = H \quad \text{for all } g \in G.$$

This is denoted as $H \triangleleft G$.

- Equivalently, a normal subgroup is one for which left and right cosets coincide.
- A normal subgroup must be a union of conjugacy classes.
- For Abelian groups, every subgroup is normal.
- The center is always a normal subgroup.
- A subgroup which contains half of all elements, i.e. for which $|G| = 2|H|$, is a normal subgroup.
- If $N \triangleleft G$, H is a subgroup of G and N is a subgroup of H , then $N \triangleleft H$. The converse is not true, i.e. if $H \triangleleft G$ and $K \triangleleft H$, it is not necessarily true that $K \triangleleft G$.

A group which has no nontrivial normal subgroups (i.e. other than $\{e\}$ and G itself) is called *simple*.

2.5 Quotient Groups

Normal subgroups allow for the construction of quotient groups. Let H be a normal subgroup of G . Loosely speaking, the quotient group G/H is obtained by dividing out the subgroup, or by considering elements of G up to the action of H . (For some stupid reason, quotient groups are also called factor groups.)

Definition 6. Let $H \triangleleft G$. The quotient group is the set of left cosets of H ,

$$G/H = \{gH \mid g \in G\}, .$$

The group operation is defined by

$$g_1H \cdot g_2H = g_1g_2H .$$

The group axioms seem to be satisfied rather obviously. However, there might be a problem: There are group elements $g \neq g'$ for which $gH = g'H$, and we have to make sure that replacing g with g' leads to the same result. This leads to the condition that H is a normal subgroup: Consider for simplicity the case where $g_1 = e$. Then $g_1H = H = hH$ for all $h \in H$. Now in G/H , H itself is the identity element, so $H \cdot gH = gH$. Hence we have

$$gH = H \cdot gH = hH \cdot gH = hgH = gg^{-1}hgH . \quad (2.2)$$

Hence we require $g^{-1}hgH = H$, or $g^{-1}hg \in H$, i.e. H needs to be normal. It is slightly more tedious, but still straightforward, to show that the product is well-defined for arbitrary elements. The key observation is that if $gH = g'H$, then $g' = gh$ with $h \in H$.

The question of well-definedness will come up now and then when one discusses quotient groups. This is a manifestation of a more general point: It is clear that $gH = g'H$ defines an equivalence relation $g \sim g'$ (though a different one as the one we used to define conjugacy classes!), and one can think of the quotient group as the set of equivalence classes under this relation, and the product is $[g_1][g_2] = [g_1g_2]$. However, this defines an operation on classes via explicit representatives, and one has to take care that the result does not depend on the chosen representative.

2.6 Group Homomorphisms

Definition 7. A group homomorphism is a map $f : G \rightarrow H$ between two groups which preserves the group structure, i.e. which satisfies

$$f(g_1g_2) = f(g_1) f(g_2) .$$

(Note that the product on the left side is the group product of G , while on the right side it is that of H .)

A group isomorphism is an invertible group homomorphism. If there exists a group isomorphism between G and H , the groups are called isomorphic, $G \cong H$.

Finally, a group isomorphism from a group to itself is a group automorphism.

We will drop the “group” and denote “group homomorphism” by “homomorphism” from now on (and similar for isomorphisms).

From the defining property of a homomorphism it directly follows that

$$f(e_G) = e_H, \quad f(g^{-1}) = (f(g))^{-1} .$$

As usual, we define kernel and image:

- The image of f , denoted by $f(G)$ (sometimes also denoted by $\text{Im}f$, but we will mostly not use that convention because of the confusability with the imaginary part), is the part of H reached by f :

$$f(G) = \{h \in H \mid \exists g \in G \text{ with } f(g) = h\}$$

- The kernel of f , denoted by $\ker f$ (or $f^{-1}(e_H)$), is the subset of G mapped to the identity in H ,

$$\ker f = \{g \in G \mid f(g) = e_H\} .$$

It is clear that f is injective if and only if $\ker f = \{e_G\}$: Assume $f(g) = f(h)$, so $f(g^{-1}h) = e_H$, i.e. $g^{-1}h \in \ker f$. If $\ker f = \{e_G\}$, then $g = h$. If, on the other hand, the kernel is nontrivial, there exists a group element $g' \neq e_G$ such that $f(g') = e_H$, and then $f(g) = f(g'g)$ while $g'g \neq g$.

From now on we also drop the G and H subscripts.

Now for three important theorems, called isomorphism theorems (sometimes the first isomorphism theorem is called homomorphism theorem):

1. Let $f : G \rightarrow H$ be a group homomorphism. Then we have the following properties:

- (a) The kernel $\ker f$ is a normal subgroup of G .
- (b) The image $f(G)$ is a subgroup of H .
- (c) The quotient $G/\ker f$ is isomorphic to $f(G)$ with the isomorphism given by

$$\tilde{f} : \begin{array}{ccc} G/\ker f & \longrightarrow & f(G) \\ g(\ker f) & \longmapsto & f(g) \end{array} . \quad (2.3)$$

2. Let H be a subgroup and N be a normal subgroup of G . Then we have

- (a) The product HN is a subgroup of G . (The product is defined as

$$HN = \{hn \mid h \in H, n \in N\} \quad (2.4)$$

- (b) The intersection $H \cap N$ is a normal subgroup of H .

(c) There is an isomorphism of quotient groups,

$$HN/N \cong H/(H \cap N). \quad (2.5)$$

3. Let H and N be normal subgroups of G , and let N be a subgroup of H . Then N is also normal in H , and

$$(G/N)/(H/N) \cong G/H. \quad (2.6)$$

Note that the second and third isomorphism theorems look intriguingly like rules for “fractions” G/N : The second theorem tells us that we can “multiply” numerator and denominator by $H \cap$,

$$\frac{HN}{N} = \frac{H \cap}{H \cap} \times \frac{HN}{N} = \frac{H \cap (HN)}{H \cap N} = \frac{H}{H \cap N}. \quad (2.7)$$

The third theorem seems to tell us that we can “cancel common factors”,

$$\frac{G/N}{H/N} = \frac{G}{N} \times \frac{N}{H} = \frac{G}{H}. \quad (2.8)$$

In these equations, the “=” and “ \times ” signs should not be taken literally – they serve as an intuitive way to memorise the theorems.

We will now prove the first isomorphism theorem. (The proofs of the other two are left to the reader – you can use the first theorem by constructing clever homomorphisms!)

(a) To show that $\ker f$ is normal, we first show that it is a subgroup. Assume $g, h \in \ker f$, i.e. $f(g) = f(h) = e$. Then clearly

$$f(gh^{-1}) = f(g)f(h^{-1}) = f(g)(f(h))^{-1} = ee^{-1} = e, \quad (2.9)$$

so $gh^{-1} \in \ker f$, thus $\ker f$ is a subgroup. To show that $\ker f$ is normal, let h now be any group element, not necessarily in the kernel, and consider

$$f(hgh^{-1}) = f(h)f(g)f(h^{-1}) = f(h)ef(h^{-1}) = f(h)(f(h))^{-1} = e, \quad (2.10)$$

so $h(\ker f)h^{-1} = \ker f$.

(b) The image is a subgroup because the group structure is transferred by the homomorphism. Explicitly, let $f(g)$ and $f(h)$ be two elements of $f(G)$. Then we have

$$f(g)(f(h))^{-1} = f(g)f(h^{-1}) = f(gh^{-1}) \in f(G). \quad (2.11)$$

(c) We need to show that \tilde{f} is a homomorphism and that the center is trivial.

But first, you might wonder whether \tilde{f} actually is well-defined: There can be a $g' \neq g$ for which still $g \ker f = g' \ker f$, so one has to make sure that $\tilde{f}(g \ker f) = \tilde{f}(g' \ker f)$, i.e. $f(g) = f(g')$. This is indeed satisfied: From $g \ker f = g' \ker f$ we can

conclude that $g^{-1}g' \ker f = \ker f$, hence $g^{-1}g' \in \ker f$. Thus, $f(g') = f(gg^{-1}g') = f(g)f(g^{-1}g') = f(g)$ since $f(g^{-1}g') = e$.

To show that \tilde{f} is a homomorphism, consider

$$\tilde{f}(g \ker f) \tilde{f}(h \ker f) = f(g)f(h) = f(gh) = \tilde{f}(gh \ker f) . \quad (2.12)$$

To show isomorphy, let $g \ker f$ be in the kernel of \tilde{f} , i.e. $\tilde{f}(g \ker f) = e$. However, the definition of \tilde{f} then implies that $f(g) = e$, hence $g \in \ker f$, so $g \ker f = \ker f$, which is the identity element of $G/\ker f$. Hence the kernel of \tilde{f} is trivial. Since it is surjective by construction, \tilde{f} is indeed a group isomorphism.

Since $\ker f$ measures the failure to be injective, this theorem states that we can “divide out” the kernel to obtain an isomorphism from any map. Practically, the theorem gives an easy way to check whether a given subgroup is normal: Find a homomorphism of which it is the kernel. This criterion is exhaustive: Since for any normal subgroup $N \triangleleft G$ the map

$$\pi : \begin{array}{ccc} G & \longrightarrow & G/N \\ g & \longmapsto & gN \end{array} \quad (2.13)$$

is a group homomorphism with $\ker \pi = N$, we have the following corollary: A subgroup $H \subset G$ is normal if and only if there exists a group homomorphism $f : G \rightarrow G'$ with $\ker f = H$.

Examples:

- For any matrix group over a field K , the determinant is a group homomorphism to $K^* = K \setminus \{0\}$. The kernel of this map consists of the matrices with unit determinant, which thus are normal subgroups. The quotients are isomorphic to the respective images. For example,

– for $GL(n, K)$ the map is surjective, and we have

$$SL(n, K) \triangleleft GL(n, K) \quad \text{and} \quad GL(n, K) / SL(n, K) \cong K^* .$$

– For elements of $U(n)$, the determinant is a complex number of modulus one, i.e. the image of the determinant map is the unit circle S^1 , which itself is isomorphic to $U(1)$

$$SU(n) \triangleleft U(n) \quad \text{and} \quad U(n, K) / SU(n, K) \cong U(1) .$$

This $U(1)$ can be identified with $U(1)_A$ above. Since the elements of $U(1)_A$ commute with every other element, $U(1)_A$ is itself a normal subgroup.

- Fix a group element g . Then conjugation by g

$$f_g : \begin{array}{ccc} G & \longrightarrow & G \\ h & \longmapsto & ghg^{-1} \end{array} \quad (2.14)$$

is a group automorphism, since

$$f_g(h_1h_2) = gh_1h_2g^{-1} = gh_1g^{-1}gh_2g^{-1} = f_g(h_1)f_g(h_2). \quad (2.15)$$

Any automorphism of G which can be written in this way is called *inner automorphism*, otherwise, it is obviously an outer automorphism. The set of inner automorphisms is again a group, and $g \mapsto f_g$ is a group homomorphism, i.e. $f_gf_h = f_{gh}$. The kernel of this homomorphism is the set of the elements which commute with all group elements — the center. This gives an independent proof that the center is normal.

2.7 Product Groups

We will now finally discuss ways to combine groups into bigger ones.

Given two groups $G_{1,2}$, we can define a “trivially combined” group, the direct product (or direct sum, which is the same for finitely many factors):

Definition 8. *The direct product of G_1 and G_2 is given by the set*

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}. \quad (2.16)$$

The product is defined componentwise,

$$(g_1, g_2)(g'_1, g'_2) = (g_1g'_1, g_2g'_2). \quad (2.17)$$

This construction is straightforwardly extended to finitely many group factors $G_1 \times \cdots \times G_n$.

- $G_1 \times G_2$ has normal subgroups $(g_1, 1)$ and $(1, g_2)$ which are isomorphic to (and which we identify with) G_1 and G_2 , respectively, and natural group homomorphisms (“projections”) $\pi_{1,2} : G_1 \times G_2 \rightarrow G_{1,2}$.
- These subgroups commute with each other, so every element of $G_1 \times G_2$ can be uniquely given as $(g_1, g_2) = (g_1, 1)(1, g_2)$.

If two groups are given, the direct product is rather trivial. On the other hand, it is interesting to see whether a given group can be written as a direct product of some of its subgroups. The result is the following: G is the direct product of its subgroups N_1 and N_2 if the following conditions hold:

1. N_1 and N_2 are normal subgroups,

2. they are basically disjoint, i.e. $N_1 \cap N_2 = \{e\}$, and
3. they generate the group, i.e. $G = N_1 N_2$.

considering the “commutator” $ghg^{-1}h^{-1}$, which by normalcy is $ghg^{-1}h^{-1} = h'h^{-1} = gg'^{-1}$, hence it is an element both of N_1 and of N_2 . By the second condition, it is thus e . By the third condition, any group element can be uniquely written as gh . Show that $(N_1 \times N_2)/N_1 \cong N_2!$

As an example, we have already seen that $U(n) = U(1) \times SU(n)$: Every $U(n)$ matrix can be written as a product of a matrix of the form $e^{i\phi}\mathbb{1}$ and an $SU(n)$ matrix, and these commute.

Things are more interesting if the subgroups are not both normal. As an example, consider $E(2)$, the group of affine transformations in the plane. Each group element is specified by a special orthogonal matrix O and a shift \vec{a} .

This structure is called semidirect product. Formally, we have the following definition:

Definition 9. Given two groups N and H and a homomorphism $\theta : H \rightarrow \text{Aut } N$, the semidirect product is defined as the group

$$G = N \rtimes H \{(n, h) \mid n \in N, h \in H\}$$

with the product

$$(n_1, h_1)(n_2, h_2) = (n_1\theta(h_1)n_2, h_1h_2) .$$

In other words, in the product H acts on N by θ . From this it follows that H is a subgroup and N is a normal subgroup of $N \rtimes H$.

To check whether a group can be written as a semidirect product, one can copy from the direct product case above, dropping the normality condition for one of the subgroups: G is the semidirect product of its subgroups N and H if

1. N is a normal subgroup,
2. they are basically disjoint, i.e. $N \cap H = \{e\}$, and
3. they generate the group, i.e. $G = NH$.

2.8 Summary

- The reason group theory is useful in physics is that group theory formalises the idea of symmetry.
- A group is a set of elements that can be multiplied associatively, there is an identity and every element can be inverted.
- A normal subgroup is a subgroup that is invariant under conjugation. It can be “divided out” to form the quotient group G/N .
- Group homomorphisms are maps that preserve the group structure.

Chapter 3

Representations

So far, we have discussed groups as sets of elements that can be multiplied with each other. In physical applications, we need these elements to act on various physical quantities, like e.g. $SO(3)$ rotations can be applied to vectors in space, or to quantum mechanical states $|nlm\rangle$. When a group acts on a vector space, this is called a representation, and this is what (most of) the rest of the course will be about. But first we will briefly discuss general group actions.

3.1 General Group Actions

In this section, we will define an action of a group on a general set X which may or may not be finite, discrete, compact or whatnot. The bijections of this set naturally form a group, which is sometimes called the *symmetric group of X* , $\text{Sym}(X)$. Note that if X is finite and contains n elements, this reproduces our previous definition of the symmetric group S_n .

Definition 10. *A group action is a homomorphism from G into $\text{Sym}(X)$.*

One usually writes the bijection associated to g as a left multiplication with g instead of $f_g(\cdot)$ or so, so if $g \in G$ and $x \in X$, the group action is written as

$$x \longmapsto g \cdot x. \tag{3.1}$$

This automatically satisfies $(gh) \cdot x = g \cdot (h \cdot x)$, and $e \cdot x = x$. Alternatively, one can write the action as a right action; this is equivalent. (In this construction, X sometimes is called a G -space.)

Examples:

- Some groups are defined by their action, e.g.
 - the symmetric group S_n acts on the n -element set,
 - and the Euclidean group acts on the affine space \mathbb{R}^n .

- The orthogonal group $O(n)$ acts on the $n - 1$ -dimensional unit sphere S^{n-1} .
- Any group acts on itself in two ways:
 - By left multiplication, $h \mapsto gh$ (or equivalently right multiplication),
 - and by conjugation, $h \mapsto ghg^{-1}$.

A group action might have various properties:

- The action is *faithful* if the kernel of the homomorphism is just $\{e\}$, i.e. if different group elements get mapped to different bijections of X .
- It is *transitive* if for all $x, y \in X$, there is a $g \in G$ such that $g \cdot x = y$.
- It is *free* if no nontrivial elements have fixed points, i.e. if $g \cdot x = x$, then $g = e$.
- If it is both transitive and free, it is *regular*.

The group action gives rise to some natural definitions:

- If the action is not faithful, the elements which leave all of X invariant form a subgroup

$$N = \{g \in G | g \cdot x = x \forall x\} . \quad (3.2)$$

- Fix a point $x \in X$. Then its *orbit* is the set of all images, i.e.

$$Gx = \{g \cdot x | g \in G\} . \quad (3.3)$$

Conversely, the *stabiliser* (or *little group*) is the set of all group elements that leave x invariant,

$$G_x = \{g \in G | g \cdot x = x\} . \quad (3.4)$$

Note that $Gx \subset X$ while $G_x \subset G$.

Finally, we give the *orbit–stabiliser theorem*: For a given point x , the orbit Gx is in one-to-one correspondence with the set of left cosets of the stabiliser. A bijection is given by

$$g \cdot x \longmapsto gG_x . \quad (3.5)$$

From this one can deduce that

$$|Gx| = \frac{|G|}{|G_x|} . \quad (3.6)$$

So much for general group actions – from now on we will restrict to representations, i.e. the case where X is a vector space and G acts by linear transformations.

3.2 Representations

Definition 11. A representation of a group G is a continuous homomorphism D from G to the group of automorphisms of a vector space V ,

$$D : G \longrightarrow \text{Aut } V. \quad (3.7)$$

V is called the representation space, and the dimension of the representation is the dimension of V .

- We will always take homomorphisms to be continuous without explicitly stating.
- Some related descriptions: G “acts on V (via D)”, V carries a representation of G , the elements of V “transform under the representation D ”, or in some abuse of language, “are in the representation D ”.
- In this lecture, we will usually assume that V is finite-dimensional unless otherwise stated.
- There is always the representation $D(g) = \mathbb{1}$ for all g . If $\dim V = 1$, this is called the *trivial representation*.
- The matrix groups, i.e. $GL(n, K)$ and subgroups, naturally have the representation “by themselves”, i.e. by $n \times n$ matrices acting on K^n and satisfying the defining constraints (e.g. nonzero determinant). This is loosely called the fundamental or defining representation.
- Two representations D and D' are called equivalent if they are related by a similarity transformation, i.e. if there is an operator S such that

$$SD(g)S^{-1} = D'(g) \quad (3.8)$$

for all g . Note that S does not depend on g ! Two equivalent representation can be thought of as the same representation in different bases. We will normally regard equivalent representations as being equal.

- A representation is called *faithful* if it is injective, i.e. $\ker D = \{e\}$, or in other words, if $D(g_1) \neq D(g_2)$ whenever $g_1 \neq g_2$.
- If V is equipped with a (positive definite) scalar product, D is unitary if it preserves that scalar product, i.e if

$$\langle u, v \rangle = \langle D(g)u, D(g)v \rangle \quad (3.9)$$

for all $g \in G$. (Here we assume that V is a complex vector space, as that is the most relevant case. Otherwise one could define orthogonal representations etc.)

3.3 Unitarity

In certain cases, we can restrict to unitary representations, due to the fact that *for finite groups and for compact Lie groups, all representations are equivalent to a unitary one.* We will show this for the case of finite groups by explicitly constructing the similarity transformation. Denote the representation by D . Unitarity means that $D(g^{-1}) = D(g)^\dagger$, so the failure to be unitary is reflected in the product $D(g)^\dagger D(g)$ not being the identity. For the whole representations, we sum over all elements and consider the operator

$$S = \sum_g D(g)^\dagger D(g), \quad (3.10)$$

which measures the non-unitarity of D which we want to transform away. Since S is Hermitean and positive definite, we can define a Hermitean square root matrix X with $X^2 = S$.

This is done by first diagonalising S , taking the square root of the diagonal matrix and undoing the diagonalisation procedure. S can be diagonalised by a unitary transformation, because it S is Hermitean, so we have $S = US_{\text{diag}}U^\dagger$. The entries of $S_{\text{diag}} = \text{diag}(\lambda_1^2, \dots, \lambda_d^2)$ are positive real numbers (because the $D(g)$ must be invertible, and so cannot have zero eigenvalues), so we can take the square root of each one and obtain the matrix root $\sqrt{S_{\text{diag}}} = \text{diag}(\lambda_1, \dots, \lambda_d)$, which in turn gives us $X = U\sqrt{S_{\text{diag}}}U^\dagger$. This X provides the sought-after similarity transformation! To see that, consider the representation

$$D'(g) = XD(g)X^{-1}, \quad (3.11)$$

which turns out to be unitary:

$$D'(g)^\dagger D'(g) = X^{-1}D(g)^\dagger X^2 D(g)X^{-1} = X^{-1}X^2 X^{-1} = \mathbb{1}, \quad (3.12)$$

where we have used $X^\dagger = X$ and

$$\begin{aligned} D(g)^\dagger X^2 D(g) &= D(g)^\dagger \left(\sum_h D(h)^\dagger D(h) \right) D(g) \\ &= \sum_h D(hg)^\dagger D(hg) = \sum_h D(h)^\dagger D(h) = X^2. \end{aligned} \quad (3.13)$$

Hence we have found the equivalent unitary representation. Note that if D was already unitary, then S and consequently X are proportional to the unit matrix, so $D' = D$.

As it stands, the proof only works for finite groups because otherwise the sum over g does not converge, so S does not exist in general. For compact Lie groups, however, there exists a (unique) translationally invariant measure, so we can replace $\sum_g \rightarrow \int dg$, and the integral is convergent because of compactness. Then the proof directly carries over.

For infinite and non-compact groups, on the other hand, the representations are not unitary in general. For infinite groups, this can be seen from a simple example: Consider \mathbb{Z}^* as multiplicative group and the obvious representation on \mathbb{C} via multiplication

$z \mapsto nz$. Clearly, $\langle nz, nz \rangle = n^2 |z|^2 \neq |z|^2$. Hence, this representation is not unitary, and it cannot be made unitary by a similarity transformation. For non-compact Lie groups, we can use a topological argument: The group of unitary operators on a finite-dimensional vector space is isomorphic to $U(n)$ (for complex vector spaces, $O(n)$ otherwise), and is hence compact. But there cannot be a bijective continuous map between a compact and a non-compact space, so faithful finite-dimensional representations of non-compact groups will be non-unitary. As an example, the defining representation $GL(n, \mathbb{C})$ (i.e. where the elements act on \mathbb{C}^n in the usual way) contains, among others, matrices $M = e^z \mathbb{1}$, for which $\langle Mu, Mv \rangle = |e^z|^2 \langle u, v \rangle$, i.e. which are manifestly not unitary. Note that the “faithful” condition is necessary, as can be seen from the representation of \mathbb{R} on \mathbb{C} as $D(x) = e^{ix}$.

The compactness argument does no longer apply if the scalar product is non-definite (but still non-degenerate), as is evident from the Lorentz group: The defining representation is one which is finite-dimensional and unitary with respect to the Minkowskian scalar product. The reason is that for non-definite spaces, the unitary operators do not form a compact set. We will discuss the Lorentz group in Chapter 7.

To summarise, for finite groups and for compact Lie groups, all representations are equivalent to a unitary one (and we will usually take them to be unitary from now on). For infinite groups and non-compact Lie groups, on the other hand, finite-dimensional faithful representations are never unitary. Finally, some non-compact groups may have representations which are unitary with respect to a non-definite scalar product, such as the Lorentz group.

3.4 Reducibility

An important question is whether we can “break up” a representation into smaller parts. This will be the case if there is a subspace of V which gets mapped to itself, because then the representation can be restricted to that subspace. Formally, we say that a subspace $V_1 \subset V$ is an *invariant subspace* if $D(g)V_1 \subset V_1$ for all $g \in G$.

Definition 12. *A representation D is called reducible if V contains an invariant subspace. Otherwise D is called irreducible.*

A representation is called fully reducible if V can be written as the direct sum of irreducible invariant subspaces, i.e. $V = V_1 \oplus \cdots \oplus V_p$, all the V_i are invariant and the restriction of D to each V_i is irreducible.

Recall from linear algebra that $V = V_1 \oplus V_2$ means that every vector v can be uniquely written as $v = v_1 + v_2$, with $v_i \in V_i$. Then $\dim V = \dim V_1 + \dim V_2$. (This is an unfortunate notation in the sense that the direct sum is basically the Cartesian product, $\oplus \approx \times$, while the tensor product \otimes is something entirely different.)

Assume D is reducible, i.e. there is an invariant subspace V_1 . From invertibility it follows that $D(g)V_1 = V_1$. If one chooses a basis of V such that the first d_1 basis vectors

span V_1 , the matrices of D take the block-upper-triangular form

$$D(g) = \begin{pmatrix} D_1(g) & D_{12}(g) \\ 0 & D_2(g) \end{pmatrix} \quad (3.14)$$

for all g . Here $D_1(g)$ and $D_2(g)$ denote $d_1 \times d_1$ - and $d_2 \times d_2$ -dimensional matrices (where $d_1 + d_2 = d = \dim V$), while the off-diagonal piece $D_{12}(g)$ is $d_2 \times d_1$ -dimensional. If D is fully reducible, on the other hand, $D_{12} = 0$, i.e. it can be brought to block-diagonal form,

$$D(g) = \begin{pmatrix} D_1(g) & & \\ & \ddots & \\ & & D_p(g) \end{pmatrix} = D_1(g) \oplus \cdots \oplus D_p(g). \quad (3.15)$$

Clearly the D_i are representations of G of dimension $d_i = \dim V_i$ (where $\sum d_i = d$) and are called the irreducible components of D . The d -dimensional representation D has been reduced to p representations D_i , each acting on an d_i -dimensional space V_i .

Reducibility does not imply complete reducibility, as can be seen in the following example: Consider the representation of the additive group of integers by two-dimensional matrices

$$T_p = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}. \quad (3.16)$$

Clearly, $T_p T_q = T_{p+q}$, although it might seem odd that addition is represented by matrix multiplication. This representation is reducible: The vector $e_{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of all T_p . On the other hand, the matrices are of Jordan form, so they are not diagonalisable. Equivalently, $e_{(2)}$ is not an eigenvector.

However, this cannot happen for most of our examples: *Unitary representations are always completely reducible*. To prove this, first note that D is fully reducible if for every invariant subspace V_1 , also its complement V_1^\perp is invariant. This property, on the other hand, follows from unitarity: Let $v \in V_1$, $w \in V_1^\perp$, i.e. $\langle w, v \rangle = 0$. Then we have

$$0 = \langle D(g)v, w \rangle = \langle v, D(g)^\dagger w \rangle = \langle v, D(g^{-1})w \rangle \quad (3.17)$$

for all g . But if g runs over all group elements, so does g^{-1} , so $\langle v, D(g)w \rangle$, hence $D(g)w \in V_1^\perp$. And since we have seen that all representations of finite and compact groups are unitary (up to equivalence), they are also completely reducible.

This in particular implies that the irreducible representations are the building blocks for all representations, so the main task in much of the later course will be to

- classify all possible irreducible representations (“irreps”) of a given group, and to
- find algorithms to reduce a representation to its irreducible components.

As a useful corollary, note that reducibility can be expressed in the following form: Let P denote a projection operator onto some subspace V_1 . Then V_1 is invariant (and hence D is reducible) if

$$PD(g)P = D(g)P \quad \text{for all } g \in G. \quad (3.18)$$

Furthermore, D is completely reducible if for every P that satisfies this, $\mathbb{1} - P$ also does, which in turn is equivalent to the statement that D and P commute.

3.5 Direct Sums and Tensor Products

In the last section we have seen how to reduce representations – there is also an opposite direction, sticking representations together to make bigger ones.

The first one is rather boring: Given two representations $D_{1,2}$ acting on spaces $V_{1,2}$ we can form the direct sum representation $D_1 \oplus D_2$ acting on $V_1 \oplus V_2$. For completeness: $V_1 \oplus V_2$ is the space of ordered pairs (v_1, v_2) on which the combined representation acts as

$$(v_1, v_2) \longmapsto (D_1(g)v_1, D_2(g)v_2). \quad (3.19)$$

This is obviously reducible, and already in block-diagonal form.

The other one is more interesting: Given V_1 and V_2 as above, we can form the tensor product (or direct product) $V = V_1 \otimes V_2$. This space is constructed as follows: Take bases $\{v_i\}$ and $\{w_a\}$ of V_1 and V_2 , where $i = 1, \dots, d_1 = \dim V_1$ and $a = 1, \dots, d_2 = \dim V_2$. A basis of V is then given by the set $\{v_i \otimes w_a\}$, i.e.

$$V = \{a^{ia} v_i \otimes w_a\} \quad (3.20)$$

with real or complex coefficients a^{ia} depending on whether V_1 and V_2 are real or complex spaces. Note that the \otimes symbol in the basis elements is just a formal way of writing these as ordered pairs. Here and always in life we use the *summation convention*: An index which appears once upstairs and once downstairs is implicitly summed over, e.g. $a^{ia} v_i \otimes w_a = \sum_{i=1}^{d_1} \sum_{a=1}^{d_2} a^{ia} v_i \otimes w_a$. An element of $V_1 \otimes V_2$ is thus specified by its components a^{ia} , which can be pictured as a “matrix”

$$a^{ia} = \begin{pmatrix} a^{11} & \dots & a^{1d_2} \\ \vdots & \ddots & \vdots \\ a^{d_1 1} & \dots & a^{d_1 d_2} \end{pmatrix}. \quad (3.21)$$

Clearly the dimension of $V_1 \otimes V_2$ is the product of the individual dimensions, $\dim V_1 \otimes V_2 = \dim V_1 \cdot \dim V_2$.

Note that although the basis elements of the tensor product space are products of the basis elements of the factors, a general element in $V_1 \otimes V_2$ cannot be written as a single tensor product of an element of V_1 and an element of V_2 . In quantum mechanics, this is called entanglement: In the simplest form, consider two spins which can each be

up or down, the states being labeled by $|\uparrow\rangle_{1,2}$ and $|\downarrow\rangle_{1,2}$. Then an entangled state would for example be

$$|\uparrow\rangle_1 \otimes |\uparrow\rangle_2 - |\downarrow\rangle_1 \otimes |\downarrow\rangle_2 \neq |\text{something}\rangle_1 \otimes |\text{something}\rangle_2 . \quad (3.22)$$

Now what about operators on the product space? Given two operators acting on V_1 and V_2 , we can define their tensor product via their matrix elements. Let the matrix elements be $(D_1)_i^j$ and $(D_2)_a^b$, the matrix elements of the product $D_1 \otimes D_2$ are

$$(D_1 \otimes D_2)_{ia}^{jb} = (D_1)_i^j (D_2)_a^b . \quad (3.23)$$

The product operator acts on V as

$$D_1 \otimes D_2 : a^{ia} v_i \otimes w_a \longmapsto \left((D_1 \otimes D_2)_{jb}^{ia} a^{jb} \right) v_i \otimes w_a . \quad (3.24)$$

For using operators, the double index notation might be somewhat unfamiliar. You can think of grouping them into one index, $(ia) = A$, which runs from one to $\dim V_1 \cdot \dim V_2$. Then the basis elements are $e_A = v_i \otimes w_j$, and vectors transform according to the usual rule $a^A \mapsto (D_1 \otimes D_2)_B^A a_B$.

In general, a tensor product is not irreducible, even if the factors are. As an example, consider the rotation group $SO(3)$. From your QM lecture you know that the irreducible representations of $SO(3)$ are labeled by the spin $l \in \mathbb{N}$, and that the spin- l representation is $2l+1$ -dimensional (although it might not have been stated in these terms). For a spin-one particle, for example, the possible spin orientations are labelled by $m = -1, 0, 1$, so the state is described by a three-component vector $|\psi\rangle = (|\psi_-\rangle, |\psi_0\rangle, |\psi_+\rangle)$. If you want to describe the spin state of the combined system of two spin-one particles (assume they are different), there clearly are nine possible combinations – this corresponds to the tensor product of two spin-one representations. However, the combined system can have total angular momentum $l = 0, 1$ or 2 , depending on the relative orientation. In other words, addition of angular momentum corresponds to the decomposition

$$D_{l=1} \otimes D_{l=1} = D_{l=0} \oplus D_{l=1} \oplus D_{l=2} . \quad (3.25)$$

Note that the dimensions work out: $3 \times 3 = 1 + 3 + 5$.

3.6 Schur's Lemma

Finally, there is one important property of irreducible components known as Schur's Lemma. We will first state this in a more formal intertwiner parlance and then in the familiar "commuting matrices" version.

Definition 13. *Given two representations D_1 and D_2 acting on V_1 and V_2 , an intertwiner between D_1 and D_2 is a linear operator*

$$F : V_1 \longrightarrow V_2 \quad (3.26)$$

which “commutes with G ” in the sense that

$$FD_1(g) = D_2(g)F \tag{3.27}$$

for all $g \in G$.

The existence of an intertwiner has a number of consequences. First, D_1 and D_2 are equivalent exactly if there exists an invertible intertwiner. Second, the kernel and the image of F are invariant subspaces: Assume $v \in \ker F$, i.e. $Fv = 0$. Then

$$FD_1v = D_2Fv = D_20 = 0, \tag{3.28}$$

so $D_1v \in \ker F$. On the other hand, let $w_2 = Fw_1$ be an arbitrary element of the image of F . Then from the definition we have

$$D_2w_2 = D_2Fw_1 = FD_1w_1, \tag{3.29}$$

which is again in the image of F . Now if D_1 is irreducible, the only invariant subspaces, hence the only possible kernels, are $\{0\}$ and V_1 itself, so F is either injective or zero. Similarly, if D_2 is irreducible, F is either surjective or zero. Taking these statements together, we arrive at *Schur’s Lemma: An intertwiner between two irreducible representations is either an isomorphism, in which case the representations are equivalent, or zero.*

An important special case is the one where $D_1 = D_2$. In that case, we see that F is essentially unique. More precisely, we have the following theorem, also often called Schur’s Lemma: If D is an irreducible finite-dimensional representation on a complex vector space and there is an endomorphism F of V which satisfies $FD(g) = D(g)F$ for all g , then F is a multiple of the identity, $F = \lambda\mathbb{1}$. Less formally, *a matrix which commutes with all matrices of an irreducible representation is proportional to the unit matrix.* This is again a way to test whether a representation is irreducible: If there is a matrix which commutes with the complete representation but is not the unit matrix, the representation is reducible.

To prove this, note that F has at least one eigenvector v with eigenvalue λ . (This is where we need V to be a complex vector space: A real matrix might have complex eigenvalues, and hence no real eigenvectors.) Clearly, $F - \lambda\mathbb{1}$ is also an intertwiner, and it is not an isomorphism since it annihilates v . Hence, by Schur’s Lemma, it vanishes, thus $F = \lambda\mathbb{1}$.

If D_1 and D_2 are not equal, but equivalent, we also find that the intertwiner is unique up to a constant: If there are two different intertwiners F_1 and F_2 , the composition $F_2^{-1}F_1$ is a self-intertwiner of D_1 and hence proportional to the identity, or in other words, $F_2 = \lambda F_1$.

This has a particular implication for Abelian groups: Since every element of a representation commutes with every other element (including itself, of course), every element is proportional to the unit matrix, and hence every (complex) representation is reducible unless it is one-dimensional, hence *all complex irreducible representations of*

Abelian groups are one-dimensional. Irreducible representations on real vector spaces can be two-dimensional.

As a neat aside, for real representations, the space of intertwiners can have real dimensions zero (for nonequivalent representations), one, two or four. This is a consequence of the following fact: The intertwiners form a division algebra over the field of the corresponding vector space (i.e. they can be added and multiplied, the identity map $\mathbb{1}$ is an intertwiner, and every nonzero map is invertible). The only division algebras over the real numbers are the real numbers, the complex numbers, the quaternions and the octonions. Since we are always discussing finite-dimensional representations, the intertwiners are basically matrices, i.e. associative, and hence the octonions do not arise. Hence, we are left with the reals, complexes and quaternions.

3.7 Eigenstates in Quantum Mechanics

From the preceding discussion, we can derive a property which is the cornerstone of the use of group theory in quantum mechanics: The eigenstates of the Hamiltonian for a given energy come in (usually irreducible) representations of the symmetry group.

To flesh this out a bit, consider the symmetry group G of the Hamiltonian, i.e. those operators which commute with H , and assume for the time being that it is finite or compact. We can divide up the Hilbert space into the eigenspaces of H , i.e. the subspaces corresponding to the different eigenvalues (this discussion applies to any self-adjoint operator which commutes with all elements of a representation of the group). Each such subspace carries a representation of G , since acting with any $D(g)$ does not change the eigenvalue because $[H, D(g)] = 0$. Furthermore, since the Hilbert space is a complex vector space, the representation can be fully reduced into its irreducible components in each eigenspace. In other words, the eigenstates of the Hamiltonian transform in representations of G , and states in different irreducible representations are orthogonal.

Hence, group theory tells you something about the degeneracy of eigenstates: The possible degeneracies of the Hamiltonian are restricted by the representations of the symmetry group – degeneracy comes from symmetry. Usually, one energy eigenspace carries one irreducible representation. If, on the other hand, there is more degeneracy, i.e. for a given energy there are several irreducible representations, this is called an accidental degeneracy. This can be either truly accidental, or it is an indication that you have not identified the full symmetry. As an example, consider the hydrogen atom: The eigenspaces of the Hamiltonian are labelled by the energy, or equivalently the principal quantum number n , since $E \sim -\frac{1}{n^2}$. (We restrict to bound states here.) The eigenspace with $n = 2$ then contains two representations of the $SO(3)$ of spatial rotations,

$$\mathcal{H}_{n=2} = \mathcal{H}_{n=2}^{l=0} \oplus \mathcal{H}_{n=2}^{l=1}. \quad (3.30)$$

Hence the degeneracy is partially from symmetry and partially accidental: The fact that the $|n, l, m\rangle = |2, 1, 1\rangle, |2, 1, 0\rangle$ and $|2, 1, -1\rangle$ states have the same energy is a consequence of the $SO(3)$ of spatial rotations. On the other hand, there is no obvious

reason why the $|2, 0, 0\rangle$ state should be degenerate with the other ones as well, so this would be an accidental degeneracy. However, this accident happens in a rather regular manner, so one might suspect that there is something behind it, and indeed it is: The symmetry group is actually larger because there is an additional $SO(3)$ generated by the Lenz–Runge vector (which, however, acts in a more subtle way). Under this larger symmetry group, the representations for any n are in fact irreducible.

Note that what we call hydrogen atom here is actually the first-order approximation – there are a number of further effects which contribute to the energy. Some of those, as e.g. the relativistic correction to the kinetic energy, preserve the $SO(3)$ symmetry and hence can only break the accidental degeneracy. Others, such as the spin of the nucleus, do not preserve the rotational symmetry and in the end break all the degeneracy.

3.8 Tensor Operators, Wigner–Eckart Theorem

To make this more explicit, we first consider a general state $|\psi\rangle$ in the Hilbert space. Under a symmetry transformation, the states transform as

$$|\psi\rangle \longmapsto D(g) |\psi\rangle, \quad \langle\psi| \longmapsto \langle\psi| D(g)^\dagger \quad (3.31)$$

with some reducible representation D of the symmetry group. Operators on the Hilbert space transform as

$$O \longmapsto D(g) O D(g)^\dagger, \quad (3.32)$$

so that matrix elements $\langle\psi|O|\chi\rangle$ are invariant. Here we make use of Wigner’s theorem which basically states that any symmetry transformation on a Hilbert space is either unitary or anti-unitary, and we take D to be unitary. Since the Hilbert space is usually infinite-dimensional, the representation D is rather unwieldy. However, in light of the preceding discussion, we can split D into finite-dimensional irreducible components. If we denote the set of irreducible representations by $\{D_{(\mu)}\}$, we can thus label the states which transform under $D_{(\mu)}$ by μ . For any group, the trivial representation will be $D_{(1)}$. We further need to specify the state within the representation by an integer i running from 1 to $\dim D_{(\mu)}$, and there will other labels – the physics of the state, e.g. the energy, which we collectively denote by x . Hence a state is completely characterised by

$$|\mu, i, x\rangle, \quad (3.33)$$

For the hydrogen atom, μ corresponds to l , i to m and x to n . We can choose the states orthonormal,

$$\langle\mu, i, x|\nu, j, y\rangle = \delta_\mu^\nu \delta_i^j \delta_x^y. \quad (3.34)$$

We use the convention that kets have upper and bras have lower indices. Note that the x label might contain continuous variables (e.g. the energy for scattering states),

then $\delta_x^y = \delta(x - y)$ with the Dirac δ function. Now a symmetry transformation simply reshuffles the states within an irreducible representation, but does not change μ or x . Explicitly, we have

$$|\mu, i, x\rangle \longmapsto (D_{(\mu)}(g))^i_j |\mu, j, x\rangle, \quad \langle \mu, i, x| \longmapsto \langle \mu, j, x| (D_{(\mu)}(g)^\dagger)^j_i. \quad (3.35)$$

Here the $D_{(\mu)}$'s are nice finite-dimensional matrices, rather than some infinite-dimensional operator. We also assume that a given representation always contains the exact same set of matrices rather than merely equivalent ones.

3.8.1 Tensor Operators

What is more, not only states, but also the operators come in representations of the symmetry group. Clearly, the operators on a Hilbert space form an algebra, hence in particular a vector space. You can convince yourself that the tensor transformation law (3.32) defines an (infinite-dimensional) representation of the group on the space of operators. Luckily, just as for the states, this representations is usually reducible, that is, we can find sets of tensors that transform among themselves in some irreducible representation. These are called *tensor operators* (here “tensor” roughly means “an object that transforms under a definite representation of a group”). From the group theory point of view, finding tensor operators is just reducing a representation (the only difference being that there is no scalar product defined, so one cannot speak of unitary representations).

Denote such a set of operators by $O_{(\mu)}^i$. Then they transform as

$$O_{(\mu)}^i \longmapsto D(g)O_{(\mu)}^i D(g)^\dagger = (D_{(\mu)}(g))^i_j O_{(\mu)}^j. \quad (3.36)$$

Note the different products: On the left-hand side the product is in the infinite-dimensional Hilbert space, while on the right-hand side it is in the finite-dimensional space of representation μ . In particular, any operator that commutes with the symmetry transformations (such as the Hamiltonian) satisfies $DOD^\dagger = O$, i.e. it is a tensor operator in the trivial representation.

As a nontrivial example, consider again the hydrogen atom. A set of tensor operators in the $l = 1$ representation is given by the angular momentum operators: The z -component acts on the whole Hilbert space as

$$J^z |nlm\rangle = m |nlm\rangle, \quad (3.37)$$

i.e. its matrix elements are $(J_z)^{nlm}_{n'l'm'} = m \delta_{n'}^n \delta_{l'}^l \delta_{m'}^m$. Under a rotation by $-\pi/2$ around the x axis it transforms into the y -component,

$$D(g)J_z D(g)^\dagger = (D_{(l=1)}(g))^i_j J^j = J^y, \quad (3.38)$$

so the three components J^x , J^y and J^z form a tensor operator in the defining, that is $l = 1$, representation, just as e.g. the position and momentum operators. Higher l tensor operators include electric and magnetic multipole operators.

3.8.2 Matrix elements of Tensor Operators

The important property of tensor operators is that they can change the representation of a state: The product of a tensor operator $O_{(\mu)}$ and a state in representation ν transforms as

$$O_{(\mu)}^i |\nu, k, x\rangle \longmapsto (D_{(\mu)}(g))^i_j (D_{(\nu)}(g))^k_l O_{(\mu)}^j |\nu, l, x\rangle, \quad (3.39)$$

which is the tensor product representation $D_{(\mu)} \otimes D_{(\nu)}$. This, together with the orthogonality relation (3.34), gives strong restrictions on the matrix elements of tensor operators.

Let us first consider invariant operators, i.e. operators in the trivial representation. This means $DOD^\dagger = O$, hence they commute with the symmetry transformations. We cannot directly apply Schur's lemma, because the representation D is reducible. However, O and D also commute on the subspaces spanned by $|\mu, i, x\rangle$ with $i = 1 \dots, \dim D_{(\mu)}$, which transform under an irreducible representation. Hence, we see that O must be proportional to the identity on each irreducible subspace, but the proportionality factor may depend on the irreducible representation. Furthermore, they do not change the representation. On the other hand, group theory does not tell us anything about the action on the x parameters, so we can write

$$O |\mu, i, x\rangle = \sum_y f_\mu(x, y) |\mu, i, y\rangle. \quad (3.40)$$

This implies for the matrix elements

$$\langle \nu, j, y | O | \mu, i, x \rangle = f_\mu(x, y) \delta_\nu^\mu \delta_j^i. \quad (3.41)$$

In other words, all the physics is contained in the function $f_\mu(x, y)$, while the dependence on the symmetry group is fixed.

Now for non-trivial representations: Consider a general tensor operator $O_{(\mu)}^i$ acting on a state $|\nu, k, x\rangle$. The product will be a state in the tensor product representation $D_{(\mu)} \otimes D_{(\nu)}$ (which we will also denote by $\mu \otimes \nu$),

$$O_{(\mu)}^i |\nu, k, x\rangle = \sum_y f_{\mu\nu}(x, y) |\mu \otimes \nu, ik, y\rangle. \quad (3.42)$$

If $\mu = 1$, which we use to denote the trivial representation, this reduces to Eq. (3.40). In contrast to Eq. (3.40), however, the tensor product representation is in general reducible, so to use the orthogonality relation (3.34), we have to decompose it into irreducible components. This we can do for each state in the sum individually,

$$|\mu \otimes \nu, ik, y\rangle = \sum_{\rho, n} C_{\rho n}^{\mu\nu ik} |\rho, n, y\rangle. \quad (3.43)$$

Here the numbers $C_{\rho n}^{\mu\nu ik}$ are called *Clebsch–Gordan coefficients* and measure the component of state $|\rho, n\rangle$ in the product $|\mu, i\rangle \otimes |\nu, k\rangle$. Usually the component n in representation ρ is fixed by i and k , so the $C_{\rho n}^{\mu\nu ik}$ has five free indices, and one can drop the sum

over n . (This equation fixes the Clebsch–Gordan coefficients up to a phase, there some convention needs to be imposed.)

This implies for the matrix element

$$\begin{aligned} \langle \sigma, a, z | O_{(\mu)}^i | \nu, k, x \rangle &= \sum_y f_{\mu\nu}(x, y) \sum_{\rho, n} C_{\rho n}^{\mu\nu ik} \langle \sigma, a, z | \rho, n, y \rangle \\ &= f_{\mu\nu}(x, z) C_{\rho a}^{\mu\nu ik} \end{aligned} \quad (3.44)$$

This is a very powerful result: We have split the matrix element into a product of a Clebsch–Gordan coefficient, which is a purely group-theoretical quantity, and a function $f_{\mu\nu}(x, z)$ which does not depend on i, k or a . (This function is sometimes called the reduced matrix element and written with double lines as $\langle \rho, z || O_{(\mu)} || \nu, x \rangle$.) The Clebsch–Gordan coefficients can be calculated once and for all for a given group, or looked up in tables. Then this equation can immediately tell you that a large number of matrix elements vanishes, and that there are relations between others, just from group theory, i.e. without ever calculating $f_{\mu\nu}(x, z)$.

For the particular case of $SO(3)$, Eq. (3.44) is known as the *Wigner–Eckart theorem*. In this case, the irreducible representations are labeled by $l \geq 0$ and the states within the representation is specified by m , so states are specified by a ket $|l, m\rangle$ and the tensor product of such states corresponds to the addition of angular momentum. There we know that the m values add, while the l value of the final state is not fixed, so we have

$$|l, m\rangle \otimes |l', m'\rangle = \sum_L C_{L, m+m'}^{ll' mm'} |L, m+m'\rangle . \quad (3.45)$$

We know (and will show later) that $C_{L, m+m'}^{ll' mm'} = 0$ unless $|l - l'| \leq L \leq l + l'$, and that there is no state $|L, M\rangle$ for $M > L$. For the particular case of $l = l' = 1$, we e.g. have

$$|1, 1\rangle \otimes |1, 1\rangle = |2, 2\rangle , \quad (3.46a)$$

$$|1, 1\rangle \otimes |1, 0\rangle = \sqrt{\frac{1}{2}} |1, 1\rangle + \sqrt{\frac{1}{2}} |2, 1\rangle , \quad (3.46b)$$

$$|1, 0\rangle \otimes |1, 0\rangle = -\sqrt{\frac{1}{3}} |0, 0\rangle + \sqrt{\frac{2}{3}} |2, 0\rangle , \quad (3.46c)$$

$$|1, 1\rangle \otimes |1, -1\rangle = \sqrt{\frac{1}{3}} |0, 0\rangle + \sqrt{\frac{1}{2}} |1, 0\rangle + \sqrt{\frac{1}{6}} |2, 0\rangle . \quad (3.46d)$$

The other combinations follow from these ones – they are basically equal, up to some minus signs. (The Clebsch–Gordan coefficients for l, l' up to 2 are e.g. listed in the Particle Data Book in Section 35.) The Wigner–Eckart theorem can be used e.g. to derive the selection rules for the Hydrogen atom: The decay of excited states via photon emission is mediated by a perturbed Hamiltonian $H = H_0 + H'$, and H' gives the coupling to the electromagnetic field. The perturbation can be expanded in tensor operators of arbitrary l , but the coefficients are suppressed by powers of the fine structure constant α . Since $\alpha \approx \frac{1}{137}$, the $l = 1$ term (dipole radiation) gives the dominant contribution to

the decay rate Γ . For the transition from state $|n, l, m\rangle$ to $|n', l', m'\rangle$, the Wigner–Eckart theorem then gives

$$\Gamma \sim \langle n, l, m | H_{L=1}^M | n', l', m' \rangle = f_{Ll'}(n, n') C_{lm}^{Ll'Mm'}. \quad (3.47)$$

From the knowledge of the Clebsch–Gordan coefficients, we thus immediately know that the dipole decay rate vanishes unless $m = M + m'$ and $|l' - 1| \leq l \leq l' + 1$. In particular, the $|n, l, m\rangle = |2, 0, 0\rangle$ state is metastable because it cannot decay via an $L = 1$ perturbation of the Hamiltonian. (This might look as though angular momentum is not conserved by the perturbed Hamiltonian, but this comes about because we are neglecting the angular momentum of the emitted photon. The full Hamiltonian of the combined atom-photon system is perfectly $SO(3)$ symmetric, hence the total angular momentum is conserved.)

3.9 Summary

- A representation is a map which assigns to each group element an endomorphism of some vector space.
- Of particular importance are irreducible representations, i.e. those that do not have invariant subspaces. For most groups of interest, all representations can be completely decomposed into the irreducible ones.
- Schur’s Lemma states that any intertwiner between two irreducible representations is either zero if the representations are not equivalent, or an isomorphism if they are. This implies that every operator that commutes with all the elements of an irreducible representation is a multiple of the identity.
- In quantum mechanics, the eigenspaces of the Hamiltonian carry a representation of the symmetry group.
- Just as states do, tensor operators transform in representations of the symmetry group. The matrix elements of tensor operators are strongly restricted by the Wigner–Eckart theorem.

Chapter 4

Discrete Groups

Now we turn to discrete groups. Discrete here roughly means that there is no notion of “closeness”, i.e. that you cannot speak of a group element which differs only infinitesimally from another one. Practically, this also implies that discrete groups can be parameterised by integers, such that $G = \{g_i\}$, where i takes values in (a subset of) \mathbb{Z} . Actually, we will for the most part of this section restrict ourselves to finite discrete groups, i.e. groups with a finite number of elements. Recall that the number of elements is called the *order* of the group, sometimes denoted by $|G|$. This is not to be confused with the order of a group element, which is the smallest number p such that $g^p = e$.

We have already mentioned a few examples:

- The cyclic group \mathbb{Z}_n is a group of order n which consists of powers of a single generating element for which $\theta^n = e$, $\mathbb{Z}_n = \{\theta^p\}$, $p = 0, \dots, n - 1$. This group is Abelian. It can be realised e.g. by the n -th roots of unity under multiplication or by the integers under addition modulo n . The cyclic groups show that there exist finite groups for any order.

In physics, \mathbb{Z}_n groups can e.g. appear as discrete rotational symmetries of molecules or crystal lattices.

- Lattices also have discrete translational symmetries, i.e. there is a basis of lattice vectors \mathbf{e}_i such that a translation by $n_i \mathbf{e}_i$ leaves the lattice invariant. In three dimensions, this is \mathbb{Z}^3 , an infinite discrete Abelian group.
- A non-Abelian example is D_4 , the symmetry transformations of a square. D_4 contains rotations and reflections.
- In some sense, specified in the next section, the most important discrete groups are the *symmetric groups* S_n , the groups of permutations of n things. We will discuss S_n in detail shortly.

We will now restrict to finite groups: G will generally be a discrete group of order N . The multiplication law of discrete groups can be given in terms of a multiplication table, where the rows and columns are labelled by the group elements, and the entry in

row g_1 , column g_2 is g_1g_2 , see Table 4.1. Each row and each column must contain every group element exactly once. The table is symmetric about the diagonal if and only if the group is Abelian.

Table 4.1: The multiplication table. Every row and every column is a permutation of the group itself. Often one omits the first row and column since they only repeat the group elements in original order.

	e	g_1	\cdots	g_n
e	e	g_1	\cdots	g_n
g_1	g_1	g_1^2	\cdots	g_1g_n
\vdots	\vdots		\ddots	
g_n	g_n	$g_n g_1$	\cdots	g_n^2

A nifty result for finite groups is Lagrange’s theorem: The order of any subgroup of G is a divisor of the order of G . To prove this, denote the subgroup by H . If $H = G$, the theorem is true. Otherwise, take an element $g \notin H$ and consider the coset gH . Clearly, $|gH| = |H|$, and H and gH are disjoint. If $H \cup gH$ does not contain all of G , continue this process until you arrive at a decomposition

$$G = H \cup g_1H \cup \cdots \cup g_mH. \tag{4.1}$$

Now the proof is complete: G has N elements, the set on the right hand side has $m \cdot |H|$ elements, hence $N = m \cdot |H|$. The number m is called the *index* of H in G .

Lagrange’s theorem has an interesting corollary: Since every element $g \in G$ generates a cyclic subgroup $\{g^p\}$ (note that any element of a finite group must have finite order), the order of any element must divide the order of the group. Thus, if the order of G is prime, G is a cyclic group generated by any of its elements. In other words, there is only one group of order p if p is prime.

4.1 The Symmetric Group

The symmetric groups are rather important – one reason is Cayley’s theorem which we will see shortly, and the other reason is that the representation theory of the symmetric group will be helpful when finding the representations of $SU(n)$ later. So let us now consider the symmetric groups in more detail. The symmetric group S_n is the group of all permutations of n things, or more formally, the group of bijections of an n -element set. The group multiplication is the composition of bijections. This clearly satisfies the group axioms. Simple combinatorics shows that the order of S_n is $n!$. Sometimes the symmetric group is called permutation group, but we will use “permutation group” for any group of permutations, i.e. also subgroups of S_n .

4.1.1 Cycles, Order of a Permutation

There are different ways of writing permutations. We can write this out explicitly as

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}, \tag{4.2}$$

meaning that the permutation maps element x_1 to x_{π_1} etc. However, this notation is not always convenient. Rather, one expresses the permutation in terms of *cycles*, sets of objects that are permuted cyclically. As an example, consider the permutation of eight elements

$$\pi = \begin{pmatrix} 12345678 \\ 23154768 \end{pmatrix} \quad (4.3)$$

We see that 1 gets mapped to 2, 2 gets mapped to 3 and 3 gets mapped back to 1, so the cycle is closed and is written as (123). The next cycle is (45) since the permutation just exchanges 4 and 5, and similarly, we get a cycle (67). Finally, 8 gets mapped to itself and hence constitutes a cycle all by itself. So we can write the permutation as

$$\pi = (123)(45)(67)(8). \quad (4.4)$$

Since the cycles have no elements in common, they commute, e.g. $(123)(45) = (45)(123)$. Furthermore, we can start a cycle at any element, as long as we keep the cyclic order:

$$(123) = (231) = (312) \neq (132). \quad (4.5)$$

Finally, if it is clear how many elements we are permuting, we may omit one-element cycles such as (8).

The order of a p -cycle is p . The inverse of a cycle is the cycle written backwards. If a permutation is given as a product of non-independent cycles, one also has to reverse the order of the cycles.

A two-element cycle, i.e. an interchange of two elements, is called a transposition. Since any permutation can be obtained by interchanging two elements at a time, we can write any cycle as a product of transpositions (which have elements in common, so they will not commute). This, however, is not unique, e.g.

$$(1234) = (14)(13)(12) = (12)(23)(34) = (23)(12)(23)(12)(34). \quad (4.6)$$

Note the difference between resolving a permutation into independent cycles and decomposing a cycle into non-independent transpositions! We see that not even the number of permutations is fixed. However, for a given permutation it is fixed whether it can be written as an even or odd number of permutations, and correspondingly, the permutation is called even or odd, respectively (or has even or odd parity, or sign ± 1). To see that the sign of a permutation is well-defined, consider the polynomial in n variables x_i

$$P(x_1, \dots, x_n) = \prod_{i < j} (x_i - x_j). \quad (4.7)$$

Then we can define the sign of a permutation π as

$$\text{sgn } \pi = \frac{P(x_{\pi(1)}, \dots, x_{\pi(n)})}{P(x_1, \dots, x_n)}. \quad (4.8)$$

This coincides with our previous definition: Clearly, the sign is ± 1 because the two polynomials have the same factors up to a sign. Furthermore, each transposition takes $P \rightarrow -P$. Hence, a permutation of given sign cannot be written both as an even and as an odd product of transpositions.

The example shows that (1234) is an odd permutation, and obviously this is valid for any four-cycle. More generally, any p -cycle $(12 \cdots p)$ can be written as

$$(12 \cdots p) = (1p)(1(p-1)) \cdots (12), \quad (4.9)$$

so any p -cycle is even if p is odd and vice versa. The parity of a product of independent cycles (i.e. having no elements in common) is the product of the parities of the cycles, so the even permutations (which include the identity) form a subgroup, called the alternating group A_n , while the odd ones do not.

4.1.2 Cayley's Theorem

The importance of the symmetric groups is that, in some sense, they contain all the finite groups. More precisely, Cayley's theorem states that every group G of order n is isomorphic to a subgroup of S_n . The proof is left as an exercise.

Cayley's theorem implies that there are only finitely many groups for a given order n , since the finite S_n can have only finitely many subgroups. The permutation groups obtained in this way have a special property: They are *regular*, meaning that permutations other than the identity leave no element unchanged. From this it follows that if $g \neq g'$, π_g and $\pi_{g'}$ never map an element to the same element, i.e. $\pi_g(x_i) \neq \pi_{g'}(x_i)$ for all i . Furthermore, in the cycle decomposition of a permutation in a subgroup of regular permutations, all cycles must be of the same length.

4.1.3 Conjugacy Classes

We will later see that the irreducible representations of a group are in one-to-one correspondence to the conjugacy classes. Let us therefore discuss the conjugacy classes of the symmetric group. Here it is more convenient to use the two-line notation. We take two permutations π and σ ,

$$\pi = \begin{pmatrix} 1 & \cdots & n \\ \pi(1) & \cdots & \pi(n) \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix}. \quad (4.10)$$

The elements conjugate to π is then

$$\begin{aligned} \sigma\pi\sigma^{-1} &= \begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix} \begin{pmatrix} 1 & \cdots & n \\ \pi(1) & \cdots & \pi(n) \end{pmatrix} \begin{pmatrix} \sigma(1) & \cdots & \sigma(n) \\ 1 & \cdots & n \end{pmatrix} \\ &= \begin{pmatrix} \sigma(1) & \cdots & \sigma(n) \\ \sigma(\pi(1)) & \cdots & \sigma(\pi(n)) \end{pmatrix} \end{aligned} \quad (4.11)$$

To see this, note that

$$\sigma^{-1} = \begin{pmatrix} \sigma(1) & \cdots & \sigma(n) \\ 1 & \cdots & n \end{pmatrix}, \quad \begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix} = \begin{pmatrix} \pi(1) & \cdots & \pi(n) \\ \sigma(\pi(1)) & \cdots & \sigma(\pi(n)) \end{pmatrix}. \quad (4.12)$$

In other words, the conjugate element is obtained by applying σ to both lines. The main message of Equation (4.11) is that the conjugate permutation has the same cycle structure as the original one. This can also be seen in the following way: If the k th element is part of a p -cycle, this means that $\pi^p k = k$. Then the conjugate permutation satisfies $(\sigma\pi\sigma^{-1})^p(\sigma k) = \sigma k$, i.e. the element σk is part of a p -cycle for the new permutation. This argument also shows that besides the cycle structure, nothing is preserved under conjugation since σ is an arbitrary permutation.

As an example, consider S_5 . The possible cycle structures, i.e. the conjugacy classes are

$$\begin{aligned} K_0 = \{e\}, \quad K_1 = \{(\cdot\cdot)\}, \quad K_2 = \{(\cdot\cdot)(\cdot\cdot)\}, \quad K_3 = \{(\cdot\cdot\cdot)\}, \\ K_4 = \{(\cdot\cdot)(\cdot\cdot\cdot)\}, \quad K_5 = \{(\cdot\cdot\cdot\cdot)\}, \quad K_6 = \{(\cdot\cdot\cdot\cdot\cdot)\}. \end{aligned} \quad (4.13)$$

Here, $(\cdot\cdot)$ denotes any two-cycle etc. The elements in K_0 , K_2 , K_3 and K_6 are even and form the alternating group A_5 .

For a general S_n , we can denote the cycle structure by an n -tuple ν of nonnegative integers, where each number ν_p gives the number of p -cycles. Here we also count 1-cycles. Then the ν_p have to satisfy one consistency condition,

$$\sum_p p\nu_p = n, \quad (4.14)$$

since there are n elements to be permuted altogether. A conjugacy class is formed by all elements of given cycle structure (ν) . A very useful way of generating all conjugacy classes is as follows: We can introduce a new set of n parameters λ_n , defined as

$$\lambda_p = \sum_{q=p}^n \nu_q. \quad (4.15)$$

Then the λ_p satisfy

$$\sum_p \lambda_p = n, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n. \quad (4.16)$$

This shows that the conjugacy classes are in one-to-one correspondence with the partitions of n . For example, class K_2 defined in Eq. (4.13) has one one-cycle and two two-cycles, so $\nu_1 = 1$, $\nu_2 = 2$ and $\nu_{3,4,5} = 0$. Hence the λ 's are $\lambda_{5,4,3} = 0$, $\lambda_2 = 2$ and $\lambda_1 = 3$. If one writes the cycle structure in terms of the λ_p , one usually omits trailing zeroes, so the classes in Eq. (4.13) can be denoted

$$\begin{aligned} K_0 = (5), \quad K_1 = (41), \quad K_2 = (32), \quad K_3 = (311), \\ K_4 = (221), \quad K_5 = (2111), \quad K_6 = (11111). \end{aligned} \quad (4.17)$$

Partitions like this are often given in a diagrammatical way, by *Young tableaux* or *Young diagrams*. They are made of n boxes, arranged in consecutive rows of λ_p boxes each, e.g.

$$(221) \sim \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \\ \hline \end{array} \quad (4.18)$$

The Young diagram has to satisfy the rule that rows and columns do not get longer going right or down. Note that while the row lengths give the λ 's, the columns directly show the cycle structure: This conjugacy class has one three-cycle and one two-cycle because it has one column of length three and one of lengths two.

As an aside on notation, note that we will later use Young tableaux with numbers in them. Various authors make a distinction between the empty tableau which they call diagram, and the numbered one, which they then call tableau. However, I will use both names interchangeably, and it should be clear from the context which type of diagram is meant.

4.2 Representations

4.2.1 The Regular Representation

The *regular representation* is a representation where the group acts on itself. To define this properly, let us first introduce an auxiliary concept, the *group algebra* R . This is a vector space of (formal) linear combinations¹ of group elements, i.e. of vectors like

$$v = \sum_g v_g g. \quad (4.19)$$

Addition is defined componentwise, i.e. $v + w = \sum_g (v_g + w_g)g$. We take the components v_g to be complex numbers, and this clearly is a vector space. However, it is even an algebra. (Recall that an algebra is a vector space with an extra bilinear “product” operation. Depending on what extra properties the product satisfies, one can have different types of algebra; this algebra will turn out to be a unital associative algebra.) The product is of course the group product, which extends to the algebra in the obvious way:

$$v \cdot w = \sum_{g,g'} v_g w_{g'} g g' = \sum_h \left(\sum_g v_g w_{g^{-1}h} \right) h. \quad (4.20)$$

Note that we have a natural inclusion of the group into the algebra, since $v = g$ can be regarded as a vector itself. We will not distinguish between g as a group element and g as a vector.

¹If you don't feel comfortable with “formal linear combinations”, you can think of functions on the group instead: Any function $f(g)$ defines a vector $\sum_g f(g)g$ and vice versa.

The group algebra is a $|G| = N$ -dimensional vector space, on which the group now acts in the obvious way: To a group element g we assign the endomorphism

$$D_{\text{reg}}(g) : v \longmapsto g \cdot v. \quad (4.21)$$

This is the regular representation.

We define an inner product on the group algebra in the usual way: The group elements as basis vectors are orthonormal, and the inner product is antilinear in the first and linear in the second argument. Hence,

$$\langle v, w \rangle = \sum_g v_g^* w_g. \quad (4.22)$$

The regular representation is unitary with respect to this inner product since

$$g \cdot v = \sum_h v_h g h = \sum_{h'} v_{h' g^{-1}} h', \quad (4.23)$$

i.e. $(g \cdot v)_h = v_{hg^{-1}}$, so the components are just reordered, which does not change the sum. This also shows that in the basis of group elements, the matrix form of the regular representation just consists of zeroes and ones such that each row and each column contains exactly one one.

We immediately see that the regular representation is reducible: The vector

$$V = \sum_g g \quad (4.24)$$

defines a one-dimensional invariant subspace, on which D_{reg} acts as the trivial representation, $D_{\text{reg}}(g)V = V$. The orthogonal space is not irreducible either (assuming $N > 2$) – if one considers e.g. the symmetric group, there is another one-dimensional subspace generated by

$$A = \sum_{\pi} \text{sgn } \pi \pi = \sum_{\pi \text{ even}} \pi - \sum_{\pi \text{ odd}} \pi \quad (4.25)$$

on which the representation acts as multiplication with the sign, $D_{\text{reg}}(\pi)|_{\langle A \rangle} = \text{sgn } \pi$. We will later see that the regular representation actually contains all irreducible representations.

4.2.2 Orthogonality Relations

Let $D_{(\mu)}$ and $D_{(\nu)}$ be two irreducible representations of G of dimensions d_{μ} and d_{ν} , acting on spaces V_{μ} and V_{ν} . (*Note the change of notation here – in the lecture I had used n_{μ} for the dimension.*) Consider the operator

$$\left(A_{(\mu\nu)}^{(ja)} \right)_b^i = \sum_g (D_{(\mu)}(g))_j^i (D_{(\nu)}(g^{-1}))_b^a, \quad (4.26)$$

where we have taken two of the indices as labels and two as actual operator indices. This way, $A_{(\mu\nu)}$ is a family of maps from V_ν to V_μ labelled by (ja) ,

$$V_\nu \ni u^b \longmapsto \left(A_{(\mu\nu)}^{(ja)} \right)_b^i u^b = v^i \in V_\mu. \quad (4.27)$$

For the moment we suppress the indices except $(\mu\nu)$.

Now we multiply $A_{(\mu\nu)}$ by $D_{(\mu)}(g)$ from the left:

$$\begin{aligned} D_{(\mu)}(g)A_{(\mu\nu)} &= \sum_{g'} D_{(\mu)}(g)D_{(\mu)}(g')D_{(\nu)}(g'^{-1}) = \sum_{g'} D_{(\mu)}(gg')D_{(\nu)}(g'^{-1}) \\ &= \sum_h D_{(\mu)}(h)D_{(\nu)}(h^{-1}g) = \sum_h D_{(\mu)}(h)D_{(\nu)}(h^{-1})D_{(\nu)}(g) \\ &= A_{(\mu\nu)}D_{(\nu)}(g) \end{aligned} \quad (4.28)$$

Hence $A_{(\mu\nu)}$ satisfies the requirement of Schur's Lemma, so we deduce that either $\mu \neq \nu$, then $A_{(\mu\nu)} = 0$, or $\mu = \nu$, then $A^{\mu\nu} \sim \mathbb{1}$.² Hence we can choose the same set of indices for both representations, and write

$$\left(A_{(\mu\nu)}^{(kl)} \right)_j^i = \delta_{\mu\nu} \delta_j^i \lambda_\mu^{(kl)}. \quad (4.29)$$

To determine $\lambda_\mu^{(kl)}$, we take the trace in the Hilbert space V_μ . Note that this is now well-defined since $A_{(\mu\nu)}$ is an endomorphism when it is nonzero. Applying the trace to the original definition (4.26), we find

$$\mathrm{tr} \left(A_{(\mu\nu)}^{(kl)} \right)_j^i = \delta_{\mu\nu} \left(A_{(\mu\nu)}^{(kl)} \right)_i^i = \delta_{\mu\nu} \sum_g \underbrace{\left(D_{(\mu)}(g^{-1}) \right)_i^l \left(D_{(\mu)}(g) \right)_k^i}_{D_{(\mu)}(g^{-1})D_{(\mu)}(g)=\mathbb{1}} = \delta_{\mu\nu} N \delta_k^l. \quad (4.30)$$

Here N is again the order of the group. The trace of the right-hand side of Eq. (4.29) gives

$$\mathrm{tr} \delta_{\mu\nu} \delta_j^i \lambda_\mu^{(kl)} = \delta_{\mu\nu} \lambda_\mu^{(kl)} \delta_i^i = \delta_{\mu\nu} \lambda_\mu^{(kl)} d_\mu. \quad (4.31)$$

Taking this together, we have

$$\lambda_\mu^{(kl)} = \frac{N}{d_\mu} \delta_k^l, \quad (4.32)$$

and the *orthogonality theorem for representations*

$$\sum_g \left(D_{(\mu)}(g) \right)_k^i \left(D_{(\nu)}(g^{-1}) \right)_j^l = \frac{N}{d_\mu} \delta_{\mu\nu} \delta_j^i \delta_k^l. \quad (4.33)$$

²Note that here we have – as we always will – assumed that all equivalent representations are given by the same set of matrices. Otherwise, it might happen that $\mu \neq \nu$, but the representations are equivalent and $A_{(\mu\nu)}$ would be an isomorphism, but not necessarily the identity.

If the representation is unitary, we can rewrite this as

$$\sum_g (D_{(\mu)}(g))^i_k (D_{(\nu)}^*(g))^j_l = \frac{N}{d_\mu} \delta_{\mu\nu} \delta_j^i \delta_k^l. \quad (4.34)$$

This, in turn, can be restated in the following way: The group algebra elements

$$v_{\mu j}^i = \sqrt{\frac{d_\mu}{N}} \sum_g (D_{(\mu)}(g))^i_j g \quad (4.35)$$

form an orthonormal set with respect to the inner product defined above. This in particular means that they are linearly independent, which gives a restriction on the dimensions of the irreducible representations: The number of linearly independent vectors cannot be larger than the dimension of the vector space, which is N . Hence we have

$$\sum_\mu d_\mu^2 \leq N. \quad (4.36)$$

We can even show that these vectors form a basis, i.e. that every vector can be expressed as a linear combination. Specifically, we will show that the group elements, which form a basis of the group algebra, can be thus expressed. Note first that we can write a group element as

$$g = \sum_h (D_{\text{reg}}(h))^e_g h. \quad (4.37)$$

To see that this is true, recall first that any row of D_{reg} contains exactly one one, and the rest is zero, so the sum on the right-hand side actually contains only one term. It is the correct one because $(D_{\text{reg}}(h))^e_g$ is one exactly for that h for which $D_{\text{reg}}(h)e = g$, i.e. for $h = g$. Now we use that fact that the regular representation is unitary, so it is completely reducible into irreducible components, which have to be (some of) the $D_{(\mu)}$, i.e. $D_{\text{reg}}(g) = U (D_{(\mu_1)} \oplus D_{(\mu_2)} \oplus \dots) U^\dagger$. This implies that each matrix element of D_{reg} is a linear combination of matrix elements of the irreducible components,

$$(D_{\text{reg}}(h))^e_g = \sum_{\mu, i, j} c_\mu^{(geij)} (D_{(\mu)}(h))^i_j, \quad (4.38)$$

where we don't know much about the coefficients, except that they are independent of h . However, this is all we need, because it implies that the g s are linear combinations of the $v_{\mu j}^i$ defined above:

$$g = \sum_h (D_{\text{reg}}(h))^e_g h = \sum_h \sum_{\mu, i, j} c_\mu^{(geij)} (D_{(\mu)}(h))^i_j h = \sum_{\mu, i, j} c_\mu^{(heij)} \sqrt{\frac{N}{d_\mu}} v_{\mu j}^i. \quad (4.39)$$

In conclusion, the basis of group elements g , and hence every element of the group algebra, can be expressed in terms of the vectors $v_{\mu j}^i$, which are furthermore linearly independent, so they form a basis. This makes the inequality in Eq. (4.36) strict:

$$\sum_{\mu} d_{\mu}^2 = N. \quad (4.40)$$

Note that this in particular implies that the number of irreducible representations is finite. Furthermore, since the $v_{\mu j}^i$ are linearly independent, they all appear in the regular representation, or in other words: The regular representation contains all irreducible representations as components.

4.3 Characters

For a given representation D , the *character* is defined as the trace of the representation matrix,

$$\chi(g) = \text{tr } D(g). \quad (4.41)$$

The characters are rather useful because they are easier to handle than the full representation, but still carry much of the information. They are easier because they are just numbers instead of matrices, and because the trace is invariant under similarity transformations,

$$\text{tr } D(g) = \text{tr } AD(g)A^{-1}. \quad (4.42)$$

This follows from the cyclic property of the trace, $\text{tr } AB = \text{tr } BA$. This implies that equivalent representations have the same characters, and that all elements in one conjugacy class have the same character. The neat thing is that the converse is also true, i.e. nonequivalent representations can be distinguished just by their characters. The characters will help us to reduce representations into their irreducible components.

Let us first derive an orthogonality theorem for characters. Recall the orthogonality theorem for representations, Eq. (4.33):

$$\sum_g (D_{(\mu)}(g))_k^i (D_{(\nu)}(g^{-1}))_j^l = \frac{N}{d_{\mu}} \delta_{\mu\nu} \delta_j^i \delta_k^l.$$

To get the characters, set $k = i$ and $l = j$ and sum over i and j . On the left hand side we obtain the traces which we denote by $\chi_{(\mu)}(g) = \text{tr } D_{(\mu)}(g)$, on the right hand side we have $\delta_j^i \delta_i^j = \delta_i^i = d_{\mu}$, so we find the *orthogonality theorem for characters*:

$$\sum_g \chi_{(\mu)}(g) \chi_{(\nu)}(g^{-1}) = N \delta_{\mu\nu}, \quad (4.43)$$

or for unitary representations

$$\sum_g \chi_{(\mu)}(g) \chi_{(\nu)}^*(g) = N \delta_{\mu\nu}. \quad (4.44)$$

Note that for finite groups, all representations are equivalent to unitary ones, so that is not a restriction.

Now we can use the fact that the characters are constant on the conjugacy classes. Label the classes by K_a , $a = 1, \dots, k$, where k is the number of classes, and let n_r label the number of elements on class K_r . Since the characters do not depend on the individual element within the class, we denote the character in class K_a as $\chi_{(\mu)}^a$. Then (4.44) can be rewritten as

$$\sum_a n_a \chi_{(\mu)}^a \chi_{(\nu)}^{a*} = N \delta_{\mu\nu}. \quad (4.45)$$

This implies that the vectors

$$\frac{1}{\sqrt{N}} (\sqrt{n_1} \chi_{(\mu)}^1, \sqrt{n_2} \chi_{(\mu)}^2, \dots, \sqrt{n_k} \chi_{(\mu)}^k) \quad (4.46)$$

form an orthonormal set in a k -dimensional space, labelled by the irreducible representations. Hence, $r \leq k$: The number of irreducible representations, r , is smaller or equal to the number of conjugacy classes k .

4.3.1 Finding Components of Representations

Before we go on to show that $r = k$, we will use characters to find the irreducible components of a given representation. If a representation is reducible, its character is a sum of the characters of the components, i.e. if $D = D^{(1)} \oplus D^{(2)}$, the character is $\chi(g) = \chi^{(1)}(g) + \chi^{(2)}(g)$. This follows from the invariance of the trace under basis changes – we do not have to actually transform D to block-diagonal form. (Similarly, the character of the tensor product of two representations is the product of the characters.) More generally, if

$$D = \bigoplus a^\mu D_{(\mu)}, \quad (4.47)$$

the character is

$$\chi = \sum a^\mu \chi_{(\mu)}. \quad (4.48)$$

Here the a_μ are nonnegative integers. If D itself is irreducible, also the character is called simple or irreducible, otherwise the character is called a *compound character*. We can now use the orthogonality theorem (4.45) to project out the coefficients:

$$\sum_a n_a \chi_{(\nu)}^{a*} \chi^a = \sum_\mu a^\mu \sum_a n_a \chi_{(\nu)}^{a*} \chi_{(\mu)}^a = \sum_\mu a^\mu N \delta_{\mu\nu} = N a^\nu, \quad (4.49)$$

hence the coefficients are

$$a^\nu = \frac{1}{N} \sum_a n_a \chi_{(\nu)}^{a*} \chi^a. \quad (4.50)$$

This also shows that representations which have the same characters are equivalent. Hence, two representations are equivalent if and only if they have the same characters.

The expression (4.50) for the coefficient has the structure of a scalar product in the space of conjugacy classes. If we take the product of the character (4.48) with itself, we obtain

$$\sum_a n_a \chi^{a*} \chi^a = \sum_{\mu, \nu} a^\mu a^\nu \sum_a n_a \chi_{(\mu)}^{a*} \chi_{(\nu)}^a = N \sum_\mu (a^\mu)^2. \quad (4.51)$$

This provides a simple criterion for irreducibility: An irreducible representation has one $a^\mu = 1$ and all others are zero, so if D is irreducible, the character satisfies

$$\frac{1}{N} \sum_a n_a \chi^{a*} \chi^a = 1. \quad (4.52)$$

Even if this quantity is larger than one but small, we can still gain some useful knowledge about a representation, due to the fact that it must be a sum of squares of integers. So e.g. if a representation has $\frac{1}{N} \sum_a n_a \chi_a^* \chi_a = 2$ or 3 , it must be a sum of two or three different irreducible components.

We now apply this to the regular representation. You can easily convince yourself that the character is

$$\chi_{\text{reg}}(g) = \begin{cases} N & \text{for } g = e, \\ 0 & \text{for } g \neq e. \end{cases} \quad (4.53)$$

From now on we will always denote the class of the identity by K_1 and the corresponding character by $\chi_{(\mu)}^1 = d_\mu$. (Clearly the trace of $D(e)$ always gives the dimension of the representation.) To find the coefficients of the irreducible representation in the regular one, use the projection (4.50):

$$a^\mu = \frac{1}{N} \sum_a n_a \chi_{(\nu)}^{a*} \chi_{\text{reg}}^a = \frac{1}{N} \chi_{(\nu)}^{1*} \chi_{\text{reg}}^1 = d_\mu. \quad (4.54)$$

Hence the coefficient of each irreducible representation in the regular representation is given by its dimension. Note that this is not entirely unexpected from Eq. (4.40).

4.3.2 Conjugacy Classes and Representations

Now let's show that $r = k$. To do this, recall that we have found $r \leq k$ by regarding $\chi_{(\mu)}^a$ as r vectors in an k -dimensional space and using the orthogonality theorem. We will now turn this around and develop a similar orthogonality relation, considering the characters as k vectors in a r -dimensional space. In other words, we want to arrive at an

expression of the form $\sum_{\mu} \chi_{(\mu)}^{a*} \chi_{(\mu)}^b \sim \delta^{ab}$ which would complement Eq. (4.43) to show $k \leq r$.

We first look at the regular representation. The previous section has shown that its character is given by

$$\left. \begin{array}{l} N \text{ for } a = 1 \\ 0 \text{ for } a \neq 1 \end{array} \right\} = (\chi_{\text{reg}})_a = \sum_{\mu} d_{\mu} \chi_{(\mu)}^a = \sum_{\mu} \chi_{(\mu)}^1 \chi_{(\mu)}^a \quad (4.55)$$

This shows that the character of the identity, considered as a r -component vector in the space of representation, is orthogonal to the other characters. We have to extend this to all classes.

Since the characters are constant over conjugacy classes, we have to reduce the group algebra to care only about classes. We do this by introducing the *class vectors*

$$\mathcal{K}_a = \sum_{g \in K_a} g. \quad (4.56)$$

The class vectors are invariant under conjugation, $g\mathcal{K}_a g^{-1} = \mathcal{K}_a$. The product of two class vectors again has this property since

$$g\mathcal{K}_a \mathcal{K}_b g^{-1} = g\mathcal{K}_a g^{-1} g\mathcal{K}_b g^{-1} = \mathcal{K}_a \mathcal{K}_b. \quad (4.57)$$

Thus it is itself a sum of class vectors. We can write this as

$$\mathcal{K}_a \mathcal{K}_b = \sum_c c_{abc} \mathcal{K}_c. \quad (4.58)$$

Hence the \mathcal{K}_a form an algebra themselves (the *class algebra*), fixed by the coefficients c_{abc} . Note that this algebra depends only on the group, but not on any representation. There is one thing we can say about the coefficients immediately: For a given conjugacy class K_a , there is a class $K_{a'}$ whose elements are the inverses of those in K_a , and $n_a = n_{a'}$. Note that K_a might be equal to $K_{a'}$. Then the product $\mathcal{K}_a \mathcal{K}_{a'}$ contains n_a copies of the identity, while for $b \neq a'$, the identity is not contained in $\mathcal{K}_a \mathcal{K}_b$. This means

$$c_{ab1} = \begin{cases} n_a & \text{for } b = a' \\ 0 & \text{for } b \neq a' \end{cases}. \quad (4.59)$$

Let us now transfer this to representations. In a given irreducible representation $D_{(\mu)}$, the class vectors are represented by

$$D_{(\mu)}^a = \sum_{g \in K_a} D_{(\mu)}(g). \quad (4.60)$$

They commute with all elements of the representation and hence are proportional to the identity, $D_{(\mu)}^a = \lambda_{(\mu)}^a \mathbb{1}$. To find the $\lambda_{(\mu)}^a$, we take the trace and find (no sum over a here!)

$$n_a \chi_{(\mu)}^a = \lambda_{(\mu)}^a d_{\mu} = \lambda_{(\mu)}^a \chi_{(\mu)}^1, \quad (4.61)$$

so

$$\lambda_{(\mu)}^a = \frac{n_a \chi_{(\mu)}^a}{\chi_{(\mu)}^1}. \quad (4.62)$$

Imposing the algebra (4.58), we find

$$\lambda_{(\mu)}^a \lambda_{(\mu)}^b = \sum_c c_{abc} \lambda_{(\mu)}^c, \quad (4.63)$$

and using Eq. (4.62) gives

$$n_a n_b \chi_{(\mu)}^a \chi_{(\mu)}^b = \chi_{(\mu)}^1 \sum_c c_{abc} n_c \chi_{(\mu)}^c. \quad (4.64)$$

To obtain a generalisation of Eq. (4.55), we sum over μ and use (4.55) and (4.59) to obtain

$$\sum_{\mu} \chi_{(\mu)}^a \chi_{(\mu)}^b = \sum_c c_{abc} \frac{n_c}{n_a n_b} \sum_{\mu} \chi_{(\mu)}^1 \chi_{(\mu)}^c = c_{ab1} \frac{n_1}{n_a n_b} N = \begin{cases} \frac{N}{n_b} & \text{for } b = a', \\ 0 & \text{for } b \neq a'. \end{cases} \quad (4.65)$$

This can be summarised as

$$\sum_{\mu} \chi_{(\mu)}^a \chi_{(\mu)}^b = \frac{N}{n_b} \delta^{a'b}, \quad (4.66)$$

and since $\chi_{(\mu)}^{a'} = \left(\chi_{(\mu)}^a\right)^*$, we finally find

$$\sum_{\mu} \chi_{(\mu)}^{a*} \chi_{(\mu)}^b = \frac{N}{n_b} \delta^{ab}. \quad (4.67)$$

Hence, the k vectors

$$\sqrt{\frac{n_a}{N}} (\chi_{(1)}^a, \chi_{(2)}^a, \dots, \chi_{(r)}^a) \quad (4.68)$$

form an orthonormal set in the r -dimensional space of irreducible representations, and so we conclude that $k \leq r$. Together with the discussion below Eq. (4.46), this implies that

$$r = k, \quad (4.69)$$

the number of representations is equal to the number of conjugacy classes.

The two orthogonality statements (4.45) and (4.67) can be summarised as follows: In the *character table*, Table 4.2, the rows are orthogonal and the columns are orthogonal when weighted with n_a . Furthermore, the squared length of the first column must be equal to the order of the group, and Eq. (4.64) implies that any product of two numbers in the same row must be expressible as a linear combination of the other numbers in a way that does not depend on the row.

	K_1	\cdots	K_k
$D_{(1)}$	$\chi_{(1)}^1$		$\chi_{(1)}^k$
\vdots		\ddots	
$D_{(k)}$	$\chi_{(k)}^1$		$\chi_{(k)}^k$

Table 4.2: A character table. The columns are labelled by the classes K_a , the rows by the irreducible representations $D_{(\mu)}$. The first row consists of only 1's, while the first column contains the dimensions of the representations.

4.4 Representations of the Symmetric Group

So far we have discussed general properties of representations. However, orthogonality statements and equations like (4.50) do not help us much if we don't know the irreducible representations or at least their characters. In this section we will therefore outline a systematic way to construct all irreducible representations for S_n . Unfortunately, I will not have time to prove most of the statements I make here, so this section is partially only a recipe, albeit an easy and powerful one.

In the previous section, we have shown that the number of irreducible representations is equal to the number of conjugacy classes, and in Section 4.1.3 we saw how the conjugacy classes are in one-to-one correspondence with the Young tableaux. Furthermore, we saw that each irreducible representation appears in the regular representation. Putting all this together, we will show how to find all irreps in the regular representation starting from the Young tableaux. This method involves finding projectors onto the invariant subspaces, so we first have to discuss the relation between projectors and subspaces.

4.4.1 Reducibility and Projectors

Any invariant subspace V of the group algebra is characterised by $D(g)v \in V$ whenever $v \in V$, or in the algebra language

$$v \in V \quad \Rightarrow \quad gv \in V \quad \forall g. \quad (4.70)$$

Clearly this property extends to linear combinations of group elements, that is, to all vectors in the group algebra. In algebra parlance, a subspace for which $v \in V$ implies $wv \in V$ for all w is said to be a *left ideal*³. Note that an ideal is in particular not only a vector subspace, but also a subalgebra. An ideal which does not contain nontrivial ideals is called *minimal* – this corresponds to irreducible subspaces. Since the regular representation is completely reducible, the algebra decomposes into a direct sum of minimal ideals,

$$R = \bigoplus_i V_i. \quad (4.71)$$

The V_i share only the zero vector, $V_i \cap V_j = \{0\}$ whenever $i \neq j$. Any element of R is uniquely expressible as a sum $v = \sum_i v_i$, where $v_i \in V_i$. Similarly, the regular representation splits as $D_{\text{reg}}(g) = \bigoplus_i D_i(g)$.

³A left ideal is an ideal that is invariant under multiplication from the left, but it is formed by acting with a projector from the right.

In particular, the identity element can be expanded as $e = \sum_i e_i$. Combining this with the defining property of the unit, we have

$$v = \sum_i v_i = ve = \sum_i (ve_i) . \quad (4.72)$$

Here $v_i \in V_i$, but also $ve_i \in V_i$ since e_i is in the left ideal V_i . Hence,

$$v_i = ve_i , \quad (4.73)$$

the ideals are generated by the e_i (by multiplication from the right). Note that in principle, each algebra element v generates the ideal Rv . However, if v is invertible, i.e. if there is a algebra element w such that $wv = e$, then $Rv = R$, so only non-invertible algebra elements generate nontrivial ideals. Now applying this to $v = e_i$, we see that

$$e_i = e_i^2 , \quad e_i e_j = 0 \quad \text{if } i \neq j . \quad (4.74)$$

So the e_i are orthogonal projectors. They are called *primitive* if they cannot be written as a sum of orthogonal projectors. Clearly, primitive projectors correspond to irreducible subspaces.

Hence, the irreducible representations of a finite group are in one-to-one correspondence with the primitive idempotents in the group algebra, and in the next section we will see a method that systematically provides all projection operators.

As examples, consider again the vectors

$$V = \sum_{\pi} \pi , \quad A = \sum_{\pi} \text{sgn } \pi \pi ,$$

which we introduced before. They are essentially projectors in the sense that

$$V^2 = n!V , \quad A^2 = n!A , \quad (4.75)$$

i.e. they are projectors up to normalisation, so $V/n!$ and $A/n!$ are true projectors. We will mostly not care about the normalisation in the following.

4.4.2 Young Tableaux and Young Operators

The operators V and A already hint at the general idea of finding irreducible representations – symmetrisation and antisymmetrisation. To see this more easily, it is convenient to make a clearer distinction between the algebra elements as operators and as states. If we think of the group elements as basis of the representation space, we denote them in ket notation by the result of the permutations, i.e. the permutation

$$\pi = \begin{pmatrix} 1 & \cdots & n \\ \pi(1) & \cdots & \pi(n) \end{pmatrix} \quad (4.76)$$

is written as

$$|\pi\rangle = |\pi(1)\cdots\pi(n)\rangle . \quad (4.77)$$

To make this more explicit, consider $n = 3$. We see that the space projected on by V is spanned by

$$|V\rangle = |123\rangle + |132\rangle + |231\rangle + |213\rangle + |312\rangle + |321\rangle , \quad (4.78)$$

while A projects onto

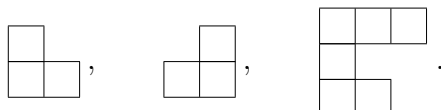
$$|A\rangle = |123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle . \quad (4.79)$$

Here, $|V\rangle$ is *totally symmetric*: It is invariant under every transposition (and hence under every permutation), such as $(12)|V\rangle = |V\rangle$. On the other hand, $|A\rangle$ is *totally antisymmetric*: Every transposition changes the sign of the state, e.g.

$$(12)|A\rangle = |213\rangle - |231\rangle + |132\rangle - |123\rangle + |321\rangle - |312\rangle = -|A\rangle . \quad (4.80)$$

It is then obvious that totally (anti)symmetric states belong to one-dimensional representations, since they can at most change sign under any transposition.

The way to construct irreducible representations of higher dimension is now to consider states of mixed symmetry, i.e. states which are symmetric under some transpositions and antisymmetric under others. The projectors on such mixed-symmetry states are constructed from the Young tableaux. For S_n , recall that a Young tableau (or Young diagram) consists of n boxes, arranged in rows and columns aligned top and left such that each row and each column does not have more boxes than the previous one. Hence these are not valid tableaux:



$$\begin{array}{c} \square \\ \square \end{array}, \quad \begin{array}{c} \square \\ \square \end{array}, \quad \begin{array}{c} \square \square \square \\ \square \square \\ \square \square \end{array} . \quad (4.81)$$

For a given tableau, one then has to write the numbers $1, \dots, n$ into the boxes. This can be done in any order. The way the Young tableau determines the symmetrisation is basically that numbers within rows are symmetrised and numbers within columns are antisymmetrised. Explicitly, this is done as follows: Symmetrisation involves all “horizontal permutations” π , which are those permutations that only permute numbers within a row, while the antisymmetry is provided by the “vertical permutations” σ which only permute numbers within columns. Their products are denoted by

$$P = \sum_{\pi} \pi , \quad Q = \sum_{\sigma} \text{sgn } \sigma \sigma , \quad (4.82)$$

and the desired projection operator is

$$Y = QP . \quad (4.83)$$

This operator (the *Young operator* associated to the tableau) projects onto an irreducible invariant subspace. Note that we first symmetrise along the rows and then antisymmetrise along the columns, so the result will in general not be symmetric anymore since Q partially destroys the effect of P .

Permutations of the numbers within a diagram lead to conjugate projectors, and to isomorphic representations. However, not all of these conjugate projectors are orthogonal – they cannot be, since there are $n!$ ways of writing the numbers, but not $n!$ copies of each representation in the group algebra. There is a “standard” rule of writing the numbers which leads to a set of orthogonal projectors, namely the requirement that within each row and column, the numbers increase going right or down. Miraculously, this rule gives exactly all the copies of any representation in the regular representation.

As an example, consider again S_3 . The three allowed tableaux are

$$\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}. \quad (4.84)$$

To turn these into projection operators, we have to write the numbers 1,2 and 3 into the boxes. For the first tableau, the only standard way is

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}. \quad (4.85)$$

Since all numbers are within one row, any permutation is horizontal, whereas the only vertical permutation is e . Hence the Young operator is

$$Y_{\square\square\square} = P = \sum_{\pi} \pi = V, \quad (4.86)$$

and this diagram corresponds to the trivial representation.

Similarly, the third tableau only admits one standard number assignment,

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}. \quad (4.87)$$

Now the rôles of horizontal and vertical are switched, and the projector is

$$Y_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} = Q = \sum_{\pi} \text{sgn } \pi \pi = A. \quad (4.88)$$

Hence, the third diagram gives the parity representation. This is basically what we could expect: Since rows and columns correspond to symmetrisation or antisymmetrisation, respectively, having all boxes in one row or one column leads to totally symmetric or antisymmetric representations. This directly extends to all the S_n .

Now let’s look at the second tableau. This admits two numberings:

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \square \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \square \\ \hline \end{array}, \quad (4.89)$$

so we have two orthogonal projectors. For the first one, the horizontal and vertical permutations are

$$P_1 = e + (12), \quad Q_1 = e - (13), \quad (4.90)$$

so the projector is

$$Y_1 = Q_1 P_1 = e + (12) - (13) - (123). \quad (4.91)$$

You can check that it is essentially idempotent, $Y^2 = 3Y$. To see which subspace it projects onto, we apply it to the basis elements:

$$\begin{aligned} Y_1 |123\rangle &= |123\rangle + |213\rangle - |321\rangle - |231\rangle = v_1, \\ Y_1 |132\rangle &= |132\rangle + |231\rangle - |312\rangle - |213\rangle = v_2, \\ Y_1 |231\rangle &= |231\rangle + |132\rangle - |213\rangle - |312\rangle = v_2, \\ Y_1 |213\rangle &= |213\rangle + |123\rangle - |231\rangle - |321\rangle = v_1, \\ Y_1 |312\rangle &= |312\rangle + |321\rangle - |132\rangle - |123\rangle = -(v_1 + v_2), \\ Y_1 |321\rangle &= |321\rangle + |312\rangle - |123\rangle - |132\rangle = -(v_1 + v_2). \end{aligned} \quad (4.92)$$

We see that this space is two-dimensional (as it should be, since we already know the representations of S_3). The second number assignment leads to a different projection operator,

$$Y_2 = Q_2 P_2 = (e - (12))(e + (13)) = e - (12) + (13) - (132). \quad (4.93)$$

You can check that this one is orthogonal to the previous one, $Y_1 Y_2 = 0$. Hence it projects onto a different subspace, which is spanned by

$$Y_2 |123\rangle = |123\rangle - |213\rangle + |321\rangle - |312\rangle = w_1, \quad (4.94)$$

$$Y_2 |132\rangle = |132\rangle - |231\rangle + |312\rangle - |321\rangle = w_2. \quad (4.95)$$

It is now clear that there can be no more orthogonal projectors, since the representation is two-dimensional and hence appears twice in the regular representation. Any other, non-standard, numbering leads to a projector which is not anymore orthogonal to the previous ones. For example,

$$\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & \\ \hline \end{array} \rightsquigarrow Y_3 = e - (12) + (23) - (123), \quad (4.96)$$

for which $Y_1 Y_3 \neq 0 \neq Y_2 Y_3$.

4.4.3 Characters, the Hook Rule

The method outlined in the previous sections systematically and explicitly gives us all irreducible representations of the symmetric groups S_n . However, for larger groups, it becomes rather tedious to work out the projectors and check the dimensions of the

associated spaces. If one is only interested in the characters, there is a simpler way: The *Frobenius formula* can give you every character in any representation. However, it is still rather involved and would take me a week to explain, so I won't do it – it is explained nicely in the book by Fulton and Harris, or the one by Hamermesh.

There is one character which is even much simpler to find – the character of the identity element, i.e. the dimension of the representation. Up to now this involved acting with the projector on every basis element and checking for linear dependence relations, but there is a faster way, called the *hook rule*: For each Young tableau, a hook is a line that comes in from the right, passes through the center of some boxes and at some box turns downward, exiting the tableau at the bottom, like this:



Hence, for each box there is one hook which turns downward at that box, and we can assign to each box the number of boxes that hook passes through. For e.g. S_{14} , a Young tableau with hook numbers would be

8	6	5	4	1	.
6	4	3	2		
5	3	2	1		
1					

(4.98)

The product of these number is the *hook factor* H of the diagram, and the dimension of the associated representation is

$$d = \frac{n!}{H}. \tag{4.99}$$

For the example above, it would be

$$d\left(\begin{array}{ccccc} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \\ \square & \square & \square & \square & \\ \square & \square & \square & \square & \end{array}\right) = \frac{14!}{8 \cdot 6 \cdot 6 \cdot 5 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2} = 21021. \tag{4.100}$$

From the hook rule it is clear that a “mirrored” diagram, i.e. one where rows and columns are interchanged, leads to the same dimensionality.

4.5 Summary

- Discrete group are those for which the elements are always finitely separated.
- A very important example is the symmetric group S_n , the group of permutations of n things.
- For finite groups, we can form the group algebra, the space of linear combinations of group elements. The group acts on the group algebra – this is the regular representation.

- For finite discrete groups, the number of irreducible representations is finite, and they are all contained in the regular representation.
- Characters carry much of the information about a representation, in particular, two representations are equivalent exactly if their characters coincide.
- For S_n , all irreducible representations can be constructed systematically using Young tableaux.

Chapter 5

Lie Groups and Lie Algebras

Now we will consider groups which are continuous, that is, groups which have a notion of “closeness”. This happens e.g. for rotations, where rotations by an angle ϕ and by $\phi + \epsilon$ are close to each other if ϵ is small. This means that the group elements will be parameterised by some real numbers (such as angles), in particular there are infinitely many group elements.

The continuous groups relevant for physics are the *Lie groups*, groups which are also manifolds. This immediately defines what we call the dimension of the group, which intuitively determines how many “independent” transformations there are. We will then see that actually the manifolds of Lie groups are not so very interesting, because they are very restricted and almost all the information is contained in the tangent space at a point, the *Lie algebra*, which corresponds to “infinitesimal group elements”. In the next chapter we will then extensively study Lie algebras and their representations.

5.1 Definition, Examples

Definition 14. *A Lie group is a group G which also has the structure of a differentiable manifold. The group and manifold structure are required to be compatible in the sense that product and inverses are continuous maps. This can be combined in the requirement that the map*

$$\begin{aligned} G \times G &\longrightarrow G \\ (g_1, g_2) &\longmapsto g_1 g_2^{-1} \end{aligned} \tag{5.1}$$

is continuous. The dimension d of the group G is the dimension of the manifold.

Recall that a d -dimensional manifold is a generalisation of the notion of a surface. We will encounter them in two types: The more intuitive concept is a *submanifold*, which is a subset of \mathbb{R}^m specified by some constraint equations. (One can also define complex manifolds by replacing \mathbb{R} with \mathbb{C} and “differentiable” with “holomorphic” in the following. However, we will only consider real manifolds, i.e. groups with real parameters. Otherwise we will make no distinction between real or complex functions, matrices etc.

and denote both \mathbb{R} and \mathbb{C} by \mathbb{K} .) The simplest example is the n -sphere S^n , which as the subspace of \mathbb{R}^{n+1} satisfying $|\vec{x}|^2 = 1$. The dimension of a submanifold is the dimension of the embedding space minus the number of constraints (if the constraints fulfill a certain regularity condition). This notion is easy to visualise, and one can denote each point on the manifold simply by its coordinate \vec{x} in the embedding space.

However, one is actually interested just in the manifold, without discussing the ambient space. Thus, to isolate the properties of the manifold, one can choose coordinates which cover only the manifold and not the full \mathbb{R}^m . For the two-sphere, one such coordinate system is the familiar longitude/latitude system used for coordinates on earth. We can then express the \vec{x} values as a function of the coordinates (this is an *embedding*), but we don't have to – indeed we usually don't. Rather, we take the coordinates to define the manifold. Generically we cannot cover the full manifold with one set of coordinates – for the sphere, the latitude/longitude coordinates miss the poles and the half-circle between them at the longitude $\varphi = 180^\circ$, because we require that the coordinates come from an open set and the map from the manifold to the coordinates is bijective. The poles are missed because they have more than one coordinate, the north pole e.g. is defined by $\vartheta = 90^\circ$ and any longitude (in the usual earth coordinates, where latitude ranges from 90° to -90° , and the equator is at $\vartheta = 0^\circ$ – for the mathematical sphere one usually takes the “polar angle” θ to range from 0 to π radians). The half-circle between the poles is missed because we can approach it from both sides with the open set, but since that cannot overlap itself because of bijectivity, there is one line it cannot reach. Topologically, this is also clear because the sphere is compact while an open set is not, so there cannot be a continuous bijection between them. However, we reach all but a set of measure zero, so for most practical applications, one set of coordinates is enough.

To formalise this a bit, an abstract manifold (without embedding space) is defined by an *atlas*: We cover G by open sets U_i and define a set of charts to \mathbb{R}^d , i.e. invertible maps

$$\phi_i : G \supset U_i \longrightarrow V_i \subset \mathbb{R}^d. \quad (5.2)$$

Since we usually cannot cover the manifold with a single such map, we have to require that on the overlap of the open sets $U_{ij} = U_i \cap U_j$, the *change of coordinates* is differentiable, i.e. the maps

$$\phi_j \circ \phi_i^{-1} : V_i \longrightarrow V_j \quad (5.3)$$

are differentiable bijections. Differentiability of functions on the manifold is defined via the charts, e.g. functions $f : \mathbb{K} \rightarrow G$ or $g : G \rightarrow \mathbb{K}^m$ are differentiable if the composed maps

$$\phi_i \circ f : \mathbb{K} \longrightarrow V_i \quad \text{and} \quad g \circ \phi_i^{-1} : V_i \longrightarrow \mathbb{K}^m \quad (5.4)$$

are differentiable (for g , this is required for all i , for f it is only necessary for those i for which U_i intersects the image of f).

For general manifolds, the maps ϕ_i^{-1} are called coordinates, i.e. a group element g is said to have coordinates $\phi_i^{-1}(g)$, and the atlas defines a coordinate system. For group, we usually use the names parameters and parameterisation.

So in the end, the definition says the group elements are parameterised by a set of d real numbers α^a , such that none are redundant. Furthermore, for two elements $g_1 = g(\alpha^a)$ and $g_2 = g(\beta^a)$, the inverse and product are also parameterised such,

$$g_1^{-1} = (g(\alpha^a))^{-1} = g(\tilde{\alpha}^a) , \quad (5.5)$$

$$g_1 g_2 = g(\alpha^a) g(\beta^a) = g(\gamma^a) , \quad (5.6)$$

and the $\tilde{\alpha}^a$ and γ^a are differentiable functions of α^a and β^a .

Notes:

- In principle, one can consider C^n -Lie groups, i.e. groups where the manifold and the group operations are n -fold continuously differentiable. However, that is not very interesting: Once a Lie group is continuous, it is already smooth, and if the manifold is finite-dimensional (which we always assume), it is analytic (Hilbert's fifth problem).
- This carries over to Lie group homomorphisms: Continuity implies smoothness, and for finite-dimensional groups even analyticity. We always assume group homomorphisms, and hence representations, to be continuous.
- Lie groups admit a unique translationally invariant measure (the Haar measure), so integration over the group is well-defined.
- We will generically choose the identity element to correspond to the origin in parameter space, i.e. $e = g(0)$.

5.1.1 Examples, Topology

The topological properties of the group are those of the underlying manifold: Groups are said to be compact, connected, simply connected etc. if the manifold is. Examples:

- For connectedness, we already saw that the $O(n)$ are not connected, since the determinant of the group elements can take only two values, ± 1 . So $O(n)$ consists of two disjoint connected components: $SO(n)$, the matrices with determinant one, and the component with determinant -1 . The $U(n)$ groups, on the other hand, are connected (the argument does not apply since the determinant can take any complex value $e^{i\phi}$).
- $SO(2) \cong U(1)$ is connected, but not simply connected: Any $R \in SO(2)$ can be written as

$$R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} , \quad a^2 + b^2 = 1 . \quad (5.7)$$

Thus the manifold is the unit circle S^1 , which is not simply connected. We can parameterise S^1 by an angle φ to obtain the usual parameterisation of $SO(2)$,

$$R(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}. \quad (5.8)$$

On the other hand, elements of $SU(2)$ can be written as

$$U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \text{with } |\alpha|^2 + |\beta|^2 = 1, \quad (5.9)$$

so they form the three-dimensional unit sphere S^3 , which is simply connected. Interestingly, these are the only spheres that are also Lie groups, due to strong restrictions on the topology of the manifold.

Recall that a space is *simply connected* if every closed curve (a loop) can be contracted to a point. Clearly, this is not true for a curve that wraps around S^1 .

- A general (topological) space is compact if each open cover contains a finite cover. This is a rather abstract (though important) notion. Luckily, for subsets of \mathbb{R}^n , there is a simpler criterion: They are compact if and only if they are closed and bounded.

Clearly, $SO(2)$ and $SU(2)$ are compact (note that we didn't need to introduce parameters for $SU(2)$ to see this). A non-compact example would be $SO(1, 1)$, the Lorentz group in two dimensions. It is defined as the group of linear transformations of \mathbb{R}^2 which leave the indefinite inner product

$$\langle \vec{v}, \vec{u} \rangle = v_1 u_1 - v_2 u_2 \quad (5.10)$$

invariant, and have determinant one. It can be written similarly to $SO(2)$ as

$$\Lambda = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad a^2 - b^2 = 1, \quad (5.11)$$

and parameterised by $\chi \in \mathbb{R}$ as

$$\Lambda(\chi) = \begin{pmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{pmatrix}. \quad (5.12)$$

Hence, as a manifold, $SO(1, 1) \cong \mathbb{R}$. Actually, since $\Lambda(\chi)\Lambda(\xi) = \Lambda(\chi + \xi)$, this isomorphism holds for the groups as well.

As another example for non-compact groups, consider $GL(n, \mathbb{K})$, which is neither bounded nor closed.

In this chapter we will generically only consider compact Lie groups.

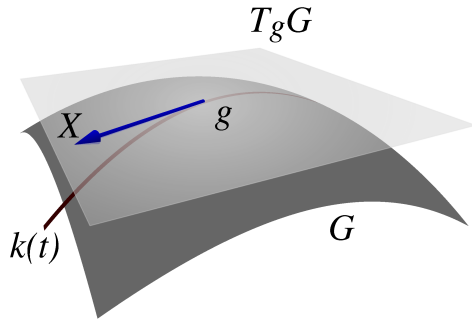


Figure 5.1: The tangent space $T_g G$ at point g of the manifold G . The tangent vector X is generated by the curve $k(t)$.

5.2 Some Differential Geometry

Since Lie groups are analytic manifolds, we can apply the apparatus of differential geometry. In particular, it will turn out that almost all information about the Lie group is contained in its tangent space at the identity, the *Lie algebra*.

Intuitively, the tangent space is just that: The space of all tangent vectors, i.e. all possible “directions” at a given point. When considering submanifolds, the tangent space can be visualised as a plane touching the manifold at the point g , see Fig. 5.1. Mathematically, the notion of a tangent vector is formalised as a differential operator – this makes intuitive sense since a tangent vector corresponds to “going” into a particular direction with a certain “speed”, i.e. the length of the vector, you notice that you move because things around you change. Hence it is reasonable that tangent vectors measure changes, i.e. they are derivatives.

5.2.1 Tangent Vectors

We now have to introduce a bit of machinery: A *curve* is a differentiable map

$$k : \mathbb{R} \supset I \longrightarrow G, \quad (5.13)$$

where I is some open interval. (Note that the map itself is the curve, not just the image.)

Definition 15. Let $k : (-\varepsilon, \varepsilon) \rightarrow G$ be a curve with $k(0) = g$. The tangent vector of k in g is the operator that maps each differentiable function $f : G \rightarrow \mathbb{K}$ to its directional derivative along k ,

$$X : f \longmapsto X[f] = \left. \frac{d}{dt} f(k(t)) \right|_{t=0}. \quad (5.14)$$

The set of all tangent vectors in g is called the tangent space of G in g , $T_g G$. This is naturally a vector space: For two tangent vectors X and Y and a real number λ , define the sum and multiple by

$$(X + Y)[f] = X[f] + Y[f], \quad (5.15)$$

$$(\lambda X)[f] = \lambda X[f]. \quad (5.16)$$

One can find curves that realise the vectors on the right-hand side, but we only care about the vectors.

Tangent vectors are defined independently of coordinates. Practically, one often needs to calculate a tangent vector in a given coordinate system, i.e. a particular map ϕ_i . Then we have

$$X[f] = \left. \frac{d}{dt} (f \circ k(t)) \right|_{t=0} = \left. \frac{d}{dt} (f \circ \phi_i^{-1} \circ \phi_i \circ k(t)) \right|_{t=0} = d(f \circ \phi_i^{-1})|_g \cdot \left. \frac{d}{dt} \phi_i(k(t)) \right|_{t=0} \quad (5.17)$$

Even more practically: if the elements of V_i , i.e. the coordinates around g , are given by x^a , then it is a common abuse of notation to write the curve as $\phi(k(t)) = x^a(t)$ and the function $(f \circ \phi_i^{-1})(x^a) = f(\phi_i^{-1}(x))$ as $f(x)$. Thus we get

$$X[f] = \frac{\partial}{\partial x^a} f(x) \cdot \left. \frac{d}{dt} x^a(t) \right|_{t=0}. \quad (5.18)$$

Here we again use the summation convention: An index that appears twice (the a) is summed over. The nice thing about this way of writing the tangent vector is that we have separated the f -dependent and k -dependent pieces, and we can even write the tangent vector without referring to f as the differential operator

$$X = \left. \frac{d}{dt} x^a(t) \right|_{t=0} \cdot \frac{\partial}{\partial x^a} = X^a \partial_a \quad (5.19)$$

Hence, the partial derivatives along the coordinate directions provide a basis for the tangent space at any given point, called the coordinate basis. Clearly, the dimension of the tangent space is equal to the dimension of the manifold. The X^a are called the components of X . This way of writing a vector comes at the price of introducing a coordinate system, and the components of the vector will depend on the chosen coordinates (as it should be: components depend on the basis). However, so do the partial derivatives, and the vector itself is entirely independent of the coordinates. Hence one often speaks of “the vector X^a ”.

5.2.2 Vector Fields

So far we have considered vectors at a given point. The next step is to extend this to the whole manifold is the notion of a vector field, roughly speaking a vector-valued function on G .

Definition 16. *A vector field is a map that associates a vector $X(g) \in T_g G$ to each point $g \in G$.*

In a given map we can choose the coordinate basis and write the components as functions of the coordinates, i.e.

$$X = X^a(x) \partial_a. \quad (5.20)$$

Clearly, the vector fields form a vector space (over \mathbb{R}) themselves. In contrast to the tangent space at a point, which is a d -dimensional vector space, however, the space of vector fields is infinite-dimensional, since every component is a function on G .

The vector fields do not only form a vector space, but an algebra. However, one cannot simply act with one vector field on another because the result would not be a first-order differential operator, so the product will be more sophisticated:

Definition 17. *Given two vector fields X and Y , the Lie bracket is a vector field given by*

$$[X, Y][f] = X[Y[f]] - Y[X[f]] . \quad (5.21)$$

This is a reflection of the fact that derivatives on manifolds are not directly straightforward. There are tangent vectors, which a priori can only act on (scalar) functions on the manifold, but not on vectors. The Lie bracket allows to extend the action to vector fields. The Lie bracket is thus sometimes called a Lie derivative, $\mathcal{L}_X Y = [X, Y]$. This is not anymore truly a directional derivative as it was for functions: It depends not only on X at the point g . To see this, observe that for any function $f : G \rightarrow \mathbb{K}$, we can define a new vector field $X' = fX$. Assume that $f(g) = 1$, so that $X'|_g = X|_g$. Then one could expect that at g also the derivatives coincide, but actually we have $\mathcal{L}_{X'} Y = f\mathcal{L}_X Y - Y[f]X$, and the second term does not vanish in general. To define “more standard” derivatives, one has to introduce a connection (or covariant derivative) – this is what happens in gauge theories and in general relativity.

The vector fields with the Lie bracket as product now form an algebra, which is different from the group algebra we encountered in Chapter 4 – it is a Lie algebra:

Definition 18. *A Lie algebra is a vector space over \mathbb{K} with a product $[\cdot, \cdot]$ (the Lie bracket) satisfying the following properties:*

- *Linearity:* $[aX, Y] = a[X, Y]$ and $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$,
- *antisymmetry* $[X, Y] = -[Y, X]$, and
- *the Jacobi identity* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Note in particular that the algebra is neither associative (instead it has the Jacobi identity, which can be understood as the Leibniz rule for Lie derivatives) nor unital (there cannot be a unit element, because antisymmetry implies that $[X, X] = 0$, so the “unit” would annihilate itself). Whenever we have an algebra with a “normal” (associative) product, such as for matrices, we can turn that into a Lie algebra by choosing the product to be the commutator $[A, B] = AB - BA$. Then the defining properties are obviously fulfilled.

As for groups, we can define homomorphisms between Lie algebras:

Definition 19. *A Lie algebra homomorphism between two Lie algebras A and B (over the same field) is a linear map that preserves the Lie bracket, i.e. a map*

$$f : \begin{array}{ccc} A & \longrightarrow & B \\ a & \longmapsto & f(a) \end{array} \quad (5.22)$$

such that

$$f([a, b]) = [f(a), f(b)] . \quad (5.23)$$

An invertible Lie algebra homomorphism is a Lie algebra isomorphism.

5.3 The Lie Algebra

The tangent vectors form a Lie algebra on any manifold. Now we will use the group property: We can multiply elements to form new elements. On the manifold, this corresponds to a motion:

Definition 20. Take a fixed element h . Multiplication by h defines the left translation

$$L_h : \begin{array}{ccc} G & \longrightarrow & G \\ g & \longmapsto & L_h g = hg \end{array} . \quad (5.24)$$

In coordinates, this is expressed as follows: Assume that $\phi(g) = \alpha^a$. Then left translation induces a motion $L_h : \alpha^a \mapsto \beta^a(\alpha)$, such that $\phi(hg) = \beta$. Of course, there is also the right translation, but that does not give different results up to some ordering switches.

Left translation is a bijection of G to itself. It also acts (more or less trivially) on functions on the manifold: To a function f it associates a new function $L_h f$ which is simply the old function moved along the manifold, i.e.

$$(L_h f)(hg) = f(g) . \quad (5.25)$$

Less trivial is the fact that it also induces a map on tangent vectors, the *differential map* (or push-forward)

$$dL_h : T_g G \longrightarrow T_{hg} G , \quad (5.26)$$

which similarly maps the vector X at point g to the vector $dL_h \cdot X$ at point hg defined by

$$(dL_h \cdot X)[f(hg)] = X[f(g)] . \quad (5.27)$$

This is sometimes written with a $*$ subscript as $dL_h = L_{h*}$. For maps from \mathbb{K}^d to \mathbb{K}^m , this is the familiar Jacobian (the matrix of derivatives $\partial f^a / \partial x^b$).

The differential map allows us to single out a particular kind of vector fields, namely those that are invariant under the differential maps of all left translations:

Definition 21. A vector field is called *left-invariant* if

$$X|_{hg} = dL_h \cdot X|_g \quad \text{for all } g, h \in G . \quad (5.28)$$

Note that the $X|_g$ notation means “the vector field X at point g ”, i.e. in coordinates, the components of the vector are evaluated at that point, and it acts on functions defined at g .

The definition implies that for left-invariant vector fields, the left-hand side of Eq. (5.27) is again the same field at the point hg , i.e.

$$X|_{hg}[f(hg)] = X|_g[f(g)] . \quad (5.29)$$

Hence this is a restriction of the g -dependence of the vector field X – it does not apply to a vector at a given point. In coordinates this is written as

$$\begin{aligned} dL_h \cdot X|_g &= X^a(hg) \frac{\partial}{\partial x^a(hg)} = X^a(g) \frac{\partial}{\partial x^a(g)} = X^a(g) \frac{\partial x^b(hg)}{\partial x^a(g)} \frac{\partial}{\partial x^b(hg)} \\ &= X^a(g) (dL_h)_a^b \frac{\partial}{\partial x^b(hg)} . \end{aligned} \quad (5.30)$$

Since left translation is the first time the group structure enters, the left-invariant vector fields are the vector fields that “know” about the group, as opposed to any old vector field which simply cares about the manifold structure. On a general manifold, we cannot single out vector fields in this way – we cannot even compare vectors at different points in a path-independent way.

The left-invariant vector fields form a subspace of the set of all vector fields. They also form a subalgebra, since

$$dL_h \cdot [X, Y]|_g = [dL_h \cdot X|_g, dL_h \cdot Y|_g] = [X|_{hg}, Y|_{hg}] = [X, Y]|_{hg} , \quad (5.31)$$

i.e. Lie brackets of left-invariant vector fields are again left-invariant. Hence we found a subalgebra of the algebra of all vector fields, and this subalgebra is determined by the group structure. This motivates the definition of the Lie algebra proper:

Definition 22. *The Lie algebra \mathfrak{g} of a group G is the space of left-invariant vector fields with the Lie bracket as product.*

The Lie algebra is generically denoted by the name of the group in lower case fraktur letters, e.g. the Lie algebra of $SU(n)$ is $\mathfrak{su}(n)$.

If one in particular chooses $g = e$, left-invariance implies that

$$X|_h = dL_h X|_e . \quad (5.32)$$

Hence the vector field is determined by its value at the identity, or in other words: the left-invariant vector fields are in one-to-one correspondence with the tangent vectors at the unit element (or any other element, for that matter). This has two important consequences:

1. The dimension of the space of left-invariant vector fields is equal to the dimension of the group. (Note that this is quite a reduction from the infinite-dimensional space of all vector fields! Of course, this is related to the fact that we imposed infinitely many conditions.)

2. All the information about the left-invariant vector fields is given by the tangent space at the identity – we can move this tangent space to any place on the manifold.

Note that this implies that we can choose any basis on T_eG and transport it to any other point while it remains a basis. This has two immediate consequences for the structure of the group manifold: First, it means the manifold is “parallelisable”, which in turn means that the tangent bundle is trivial. This is a fancy way of saying that the disjoint union of all tangent spaces is just the Cartesian product $G \times T_eG$, so that the components of vector fields are indeed just functions of G . This is a rather strong constraint. For example, among the spheres only S^1 , S^3 and S^7 have trivial tangent bundle. (For S^2 , you might know the “hairy ball theorem” which states that any vector field must vanish at at least one point. Thus the coefficients cannot be arbitrary functions, and the tangent bundle cannot be trivial.) Second, it means that all tangent spaces are basically the same, so also geometric properties such as curvature are the same everywhere. These properties hint at that the geometry of Lie groups is not particularly interesting, and it might be sufficient to analyse the Lie algebra. This is more or less true – we will make this precise in Section 5.5.

5.4 Matrix Groups

5.4.1 The Lie Algebra of Matrix Groups

Since most Lie groups of interest in physics are matrix groups, it is worthwhile to look at those in more detail. Let us consider $GL(n, \mathbb{K})$, since this is the most general one. As coordinates we choose the entries of the matrices, so that a matrix g is parameterised by $g = g^i_j$. In particular, the identity is $e = \delta^i_j$. Then the left translation, as multiplication, acts as

$$L_h g = hg = h^i_k g^k_j. \quad (5.33)$$

Its differential is

$$(dL_h)^i_l{}_{j k} = \frac{\partial (hg)^i_j}{\partial g^k_l} = h^i_k \delta^l_j \quad (5.34)$$

The left-invariant vector fields can be obtained from the tangent vectors at the identity. Denote such a vector by

$$V = V^i_j \left. \frac{\partial}{\partial g^i_j} \right|_{g=e}. \quad (5.35)$$

The vector field X_V corresponding to V is given by acting on V with the differential,

$$X_V|_h = dL_h V = (dL_h)^i_l{}_{j k} V^k_l \frac{\partial}{\partial h^i_j} = h^i_k \delta^l_j V^k_l \frac{\partial}{\partial h^i_j} = (hV)^i_j \frac{\partial}{\partial h^i_j}, \quad (5.36)$$

i.e. the component of X_V at the point h is just hV , interpreted as a matrix product. This gives us a very important formula for the Lie bracket: Let X_V and X_W be two vector fields obtained from tangent vectors V and W as above. The Lie bracket is a new vector field, which at point h is given by

$$\begin{aligned}
[X_V, X_W]|_h &= \left((X_V|_h)^i_j \frac{\partial}{\partial h^i_j} (X_W|_h)^k_l - (X_W|_h)^i_j \frac{\partial}{\partial h^i_j} (X_V|_h)^k_l \right) \frac{\partial}{\partial h^k_l} \\
&= \left(h^i_m V^m_j \frac{\partial}{\partial h^i_j} h^k_n W^n_l - h^i_m W^m_j \frac{\partial}{\partial h^i_j} h^k_n V^n_l \right) \frac{\partial}{\partial h^k_l} \\
&= h^k_m (V^m_j W^j_l - W^m_j V^j_l) \frac{\partial}{\partial h^k_l} \\
&= h [V, W] \frac{\partial}{\partial h}.
\end{aligned} \tag{5.37}$$

In the last line, the square brackets indicate not the Lie bracket of vector fields, but the matrix commutator! That means that we can identify the Lie algebra of $GL(n, \mathbb{C})$ with the components V^i_j of tangent vectors and use the usual matrix commutator as the product, which is a huge simplification. This is the way we will discuss Lie algebras in the following.

5.4.2 The Lie Algebra of $GL(n, \mathbb{K})$ and Subgroups

We will now quickly discuss the Lie algebras of the classical matrix groups, which all are subgroups of $GL(n, \mathbb{C})$ or $GL(n, \mathbb{R})$. Since we saw before that the Lie algebra is determined by the tangent vectors V at $g = e$, we need to derive constraints on the components V^i_j .

The general idea is as follows: Recall that tangent vectors can be defined as derivatives of curves and consider a curve $g(t)$ with $g(0) = e$. Around the identity, it can be expanded as

$$g(t) = e + tB + \mathcal{O}(t^2) \tag{5.38}$$

with some matrix B , which will be the component matrix of the tangent vector. To determine the tangent space, we have to impose the defining constraints on the group and see what that implies for B :

- For $GL(n, \mathbb{K})$, the constraint is that the determinant is nonzero. This is satisfied for any B as long as t is small enough, so there is no restriction: The Lie algebra $\mathfrak{gl}(n, \mathbb{K})$ consists of all $n \times n$ matrices with entries in \mathbb{K} .
- For $SL(n, \mathbb{K})$, the determinant must be one. Since

$$\det(\mathbb{1} + tB) = 1 + t \operatorname{tr} T, \tag{5.39}$$

we see that the tangent space contains all traceless matrices – this reduces the dimension by one. Note that tracelessness is a reasonable constraint in the sense that it is conserved under addition, scalar multiplication and taking the commutator.

- The unitary groups require that $\mathbb{K} = \mathbb{C}$. The constraint for $U(n)$, $g^\dagger g = e$, is, to first order,

$$(\mathbb{1} + tB)^\dagger (\mathbb{1} + tB) = \mathbb{1} + t(B + B^\dagger) + \mathcal{O}(t^2) . \quad (5.40)$$

Hence B must be anti-Hermitian, $B = -B^\dagger$. Note here that there are two conventions: Mathematicians work with anti-Hermitian B , while physicists usually prefer Hermitian matrices and write $B = iC$ with $C = C^\dagger$.

Matrices in $\mathfrak{su}(n)$ have to be traceless in addition, just as for $\mathfrak{sl}(n)$.

- The real variant of this is $O(n)$. By the same argument as before, B has to be antisymmetric, $B = -B^T$. This is automatically traceless – the dimensions of $O(n)$ and $SO(n)$ are the same. Since $O(n)$ is not connected, it is also clear that there is no curve from the identity to a matrix with determinant -1 , so the tangent space anywhere at $SO(n)$ does not see $O(n)$.

Again, some physicists add a factor i to deal with Hermitian matrices.

5.5 From the Algebra to the Group

It is straightforward to determine the Lie algebra of a given group, in particular it is unique. This raises the question about the converse: Given a Lie algebra (as a vector space with some bracket), is there always a Lie group, and is this group unique? And how do we find it?

5.5.1 Lie's Third Theorem

The importance of the Lie algebra is that the Lie algebra almost uniquely determines the group. To be precise, we have *Lie's third theorem*¹, which we do not prove: Let \mathfrak{g} be a Lie algebra of dimension d . Then there exists a unique connected and simply connected d -dimensional Lie group G such that its Lie algebra is (isomorphic to) \mathfrak{g} .

So we see that the group always exists, while uniqueness involves two qualifiers, which are related to the global structure of the group: The group is unique if it is both connected and simply connected. The first one is obvious: Since the Lie algebra corresponds to the tangent space at the identity, it does not know about disconnected pieces – we saw already that $\mathfrak{o}(n) = \mathfrak{so}(n)$. The second part is more interesting: The group must be simply connected. Turning this around, this means that for each non-simply connected Lie group G , there is a unique group \bar{G} with the same algebra but which is simply connected. This group is called the *universal covering group*.

As an example, consider $SO(2) \cong U(1) \cong S^1$, which is not simply connected. Its universal covering group must be one-dimensional and cannot meet back up with itself, so it must be \mathbb{R} (a one-dimensional manifold is isomorphic to either S^1 or \mathbb{R}). There

¹Strictly speaking, this is the converse of Lie's third theorem, but it is the more important statement.

is a group homomorphism from R to S^1 , $x \mapsto e^{2\pi i x}$ with kernel \mathbb{Z} , so we have the isomorphism

$$S^1 \cong \mathbb{R}/\mathbb{Z}. \quad (5.41)$$

Similarly, the algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic, but the groups are not.

What can we say in general? Let \bar{G} be the universal cover and G be a different group with the same algebra. Then there is a surjective homomorphism $f : \bar{G} \rightarrow G$. The kernel $\ker f$ is a normal subgroup, and we have the isomorphism

$$G \cong \bar{G}/\ker f. \quad (5.42)$$

What is more, $\ker f$ is discrete (since the groups have the same dimension) and hence its elements commute with all of \bar{G} , i.e. $\ker f \subset Z(\bar{G})$. This shows that every such G can be obtained from the universal covering group by dividing out a subgroup of the center.

5.5.2 The Exponential Map

We will now discuss how to go back from the algebra to the group, basically by interpreting the algebra elements as “infinitesimal group elements” and reconstructing the finite elements. To do so, first consider the group G and a curve $\phi : \mathbb{R} \rightarrow G$ which satisfies

$$\phi(s)\phi(t) = \phi(s+t). \quad (5.43)$$

Such a curve is called a *one-parameter subgroup*, because it is a subgroup which has one parameter (i.e. its image is a one-dimensional Lie group). Clearly, this subgroup is Abelian, and we have $\phi(0) = e$ and $\phi(t)^{-1} = \phi(-t)$.

The one-parameter subgroup defines a tangent vector at every $\phi(t)$ whose components are

$$X^a(\phi(t)) = \frac{d\phi^a(t)}{dt}. \quad (5.44)$$

This is not quite a vector field, since it is defined only on the curve. However, it is a left-invariant “vector field” along the curve: From Eq. (5.30) we see that the components of a left-invariant vector fields have to satisfy

$$X^a(hg) = \frac{\partial x^a(hg)}{\partial x^b(g)} X^b(g). \quad (5.45)$$

Restricting this to the curve, we take $h = \phi(t)$ and $g = \phi(s)$, hence $hg = \phi(t+s)$. From the definition (5.44), we find by the chain rule

$$X^a(hg) = X^a(\phi(t+s)) = \frac{d\phi^a(t+s)}{d(t+s)} = \frac{\partial \phi^a(t+s)}{\partial \phi^b(s)} \frac{d\phi^b(s)}{ds}, \quad (5.46)$$

hence Eq. (5.45) is satisfied. On the other hand, one can show that a left-invariant vector field uniquely defines a one-parameter subgroup (this is true basically because one has to solve the ordinary differential equation (5.44) with boundary condition $\phi(0) = e$, which has a unique solution). The proof of these statements is contained in more detail in Nakahara's book.

Hence we see that one-parameter subgroups and left-invariant vector fields are in one-to-one correspondence. Furthermore, rescaling the vector field $X \rightarrow \lambda X$ simply gives a rescaling of the parameter of the subgroup, $t \rightarrow t/\lambda$. Since $\mathfrak{g} \cong T_e G$, we can thus associate to each tangent vector V (at the identity) a one-parameter subgroup $\phi_V(t)$, and $\phi_{\lambda V}(t) = \phi_V(t/\lambda)$. This motivates the following definition:

Definition 23. For a Lie group G and a tangent vector $V \in T_e G$, the exponential map is defined as

$$\begin{aligned} \exp : T_e G &\longrightarrow G \\ V &\longmapsto \exp(V) = \phi_V(1). \end{aligned} \tag{5.47}$$

The rescaling property mentioned above translates to

$$\exp(tV) = \phi_V\left(\frac{1}{t}\right) \tag{5.48}$$

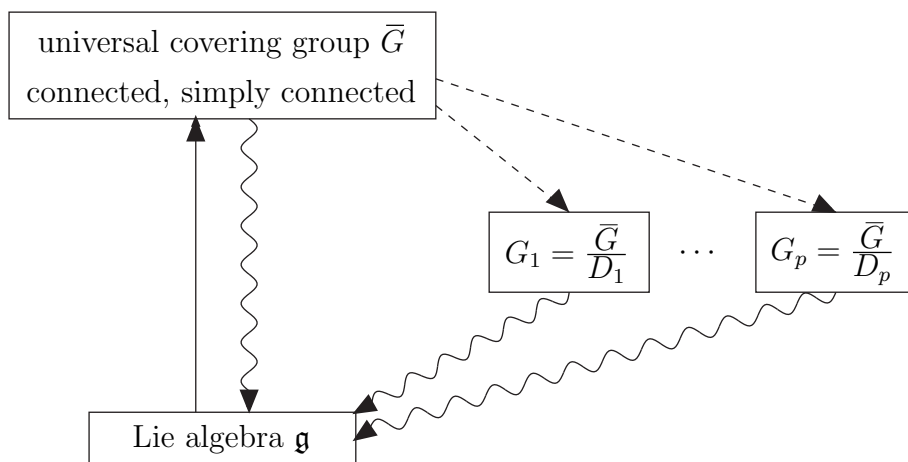


Figure 5.2: Relationship between a Lie algebra, the associated groups and the universal covering group. Wiggly lines correspond to linearisation, dashed lines to group quotients and the full line to the exponential map.

The relations between the algebra, the covering group and its quotients are summarised in Figure 5.2.

The Exponential of a Matrix

For matrix groups, the exponential map is actually the exponential of the Lie algebra elements. This can be seen by noting that the matrix exponential actually forms a one-parameter subgroup with the correct boundary condition at the identity. Hence, for the matrix Lie algebras we discussed in Section 5.4, we can express all group elements as the exponential of some algebra element. Since the exponential of a matrix might not be familiar to everybody, we will briefly discuss it here.

For any matrix A , the exponential is defined via the series expansion,

$$\exp A = e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \quad (5.49)$$

Some properties of the usual exponential of numbers carry over:

- It is by definition analytic, and the sum is absolutely convergent for any A .
- $\exp(0) = A^0 = \mathbb{1}$: It maps a neighbourhood of zero to a neighbourhood of the identity. Furthermore,

$$\frac{d}{dt} e^{tA} = A e^{tA}, \quad (5.50)$$

which is nonzero. So for small enough t , the map $t \mapsto e^{tA}$ is invertible – it might not be globally because there might be a nonzero t such that $e^{tA} = \mathbb{1}$ again.

- If the eigenvalues of A are λ_i , the eigenvalues of e^A are e^{λ_i} . This is obvious for diagonalisable A , but still holds for any matrix (which you can bring to Jordan form, where it is easy to see). This in particular implies that the eigenvalues, and hence the determinant, of e^A are never zero! What is more, we have

$$\det e^A = \prod_i e^{\lambda_i} = e^{\sum_i \lambda_i} = e^{\text{tr } A}. \quad (5.51)$$

This is a very useful formula – remember it for life!

The main difference to the exponential of numbers comes from the fact that two matrices do not commute in general, hence the product of exponentials is not simply the exponential of the sum. Rather, we have the Baker–Campbell–Hausdorff (BCH) formula,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[B,A]])+\dots}, \quad (5.52)$$

where the dots stand for higher commutators. Note that the BCH formula only involves commutators and not simple products, so it is well-defined on the Lie algebra! If A and B commute, then the exponent on the right-hand side is just $A + B$, so we in particular have (for numbers s and t)

$$e^{tA} e^{sA} = e^{(s+t)A} \quad \text{and thus} \quad (e^{tA})^{-1} = e^{-tA}. \quad (5.53)$$

As an example, consider $\mathfrak{so}(2)$. As we have seen, that consists of the real antisymmetric 2×2 matrices, so it is one-dimensional and spanned e.g. by

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (5.54)$$

To calculate the exponential, note that $T^2 = -\mathbb{1}$, so the sum splits into even and odd powers,

$$\begin{aligned} e^{\alpha T} &= \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha T)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \alpha^{2n} \cdot \mathbb{1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \alpha^{2n+1} \cdot T \\ &= \cos \alpha \mathbb{1} + \sin \alpha T = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}. \end{aligned} \quad (5.55)$$

Hence we reach all of $SO(2)$ in this way. Note that in this example the exponential map is only bijective for t in a neighbourhood of zero since $e^{2\pi T} = \mathbb{1}$ again.

As a remark, note that one can exponentiate other operators, e.g. derivatives: Assume $f(x)$ is analytic. Then the Taylor expansion is reproduced by the exponential, so we can move the function:

$$\exp\left(a \frac{d}{dx}\right) f(x) = f(x + a). \quad (5.56)$$

5.6 Summary

- Lie groups are groups which are at the same time manifolds.
- Manifolds locally look like \mathbb{R}^n , but can be rather different globally.
- Tangent vectors are differential operators at a point, vector fields are vectors that vary over the manifold.
- For any manifold, the vector fields form an infinite-dimensional Lie algebra with the Lie bracket. On Lie groups, the left translation picks out the left-invariant vector fields which form the finite-dimensional Lie algebra associated to the Lie group.
- For matrix groups, the Lie bracket of vector fields can be expressed as the matrix commutator of tangent vectors at the identity.
- The algebra specifies the group uniquely up to connectedness and simply-connectedness. The exponential map recovers the universal covering group from the algebra.

Chapter 6

Representations of Lie Algebras

As we saw in the last chapter, the Lie algebra carries (almost) as much information as the group itself. But since in the algebra, we can add and multiply, rather than only multiply, elements, the algebra is more convenient to work with. (This is also why we used the group algebra for finite groups.) Hence, the task is now to classify representations of Lie algebras. We will first make a few remarks on representations, and then discuss representations of $\mathfrak{su}(2)$ in detail before going into the more general procedure.

6.1 Generalities

6.1.1 Structure Constants

Since the algebra is a vector space, we can choose a basis T_i of \mathfrak{g} , and the T_i are called *generators* of the group. The Lie bracket is then fixed by its action on the basis elements:

$$[T_i, T_j] = f_{ij}^k T_k. \quad (6.1)$$

The numbers f_{ij}^k are called *structure constants*. There is again an issue with conventions: Since $[A, B]^\dagger = [B^\dagger, A^\dagger]$, the left-hand side is anti-Hermitian if $T_i^\dagger = \pm T_i$. Hence, if the generators are chosen Hermitian (physicists' convention), the structure constants as defined above are purely imaginary, so it is customary to include another factor of i to make them real,

$$[T'_i, T'_j] = i f'_{ij}{}^k T'_k. \quad (6.2)$$

The $f'_{ij}{}^k$ are called structure constants as well, and the prime is usually omitted, so one has to keep in mind which convention is used. For now, we will stick to anti-Hermitian generators and real structure constants, but we will change later when we discuss $SU(2)$.

The structure constants inherit two properties from the Lie bracket:

1. Antisymmetry follows from the antisymmetry of the commutator, $f_{ij}^k = -f_{ji}^k$.

2. The Jacobi identity translates to

$$f_{ij}^k f_{kl}^m + f_{jl}^k f_{ki}^m + f_{li}^k f_{kj}^m = 0. \quad (6.3)$$

Turning this around, one can define a Lie algebra by postulating d generators T_i and d^3 numbers f_{ij}^k satisfying these two properties.

6.1.2 Representations

A representation of a Lie algebra is, similarly to a group, a realisation of the algebra by matrices:

Definition 24. *A representation of a Lie algebra \mathfrak{g} is a Lie algebra homomorphism D from \mathfrak{g} to a Lie algebra of matrices (or endomorphisms of a vector space) with the matrix commutator as Lie bracket. The dimension of the representation is the dimension of the vector space.*

A representation is reducible if there is an invariant subspace.

Representations which are related by a similarity transformation are called equivalent.

Note that this implies that in a representation, the product of two generators is well-defined, since it is simply the matrix product.

A representation of the algebra \mathfrak{g} gives us – via the exponential map – a representation of the universal covering group \bar{G} . This is not necessarily true for non-simply connected groups G_i , i.e. some of the representations of \mathfrak{g} might not correspond to true representations of G_i . Reducible representations of the algebra lead to reducible representations of the group. Note that, in contrast to group representations, the endomorphisms that form a representation of the algebra are in general not invertible.

6.1.3 The Adjoint Representation

There is a representation which exists for every Lie algebra (besides the trivial one for which $D(X) = 0$ identically), called the *adjoint representation*. The idea is similar to the regular representation: The Lie algebra itself is a vector space, and the algebra acts on itself by the Lie bracket. Hence we define the adjoint representation ad by

$$\text{ad}(X)Y = [X, Y]. \quad (6.4)$$

We will usually omit the ad and just write the commutator because the adjoint action is basically the Lie algebra action itself. Clearly, the dimension of the adjoint representation is equal to the dimension of the Lie algebra d . Note that the adjoint representation is the infinitesimal version of the action of the group on itself by conjugation,

$$e^X e^Y e^{-X} = e^{Y+[X,Y]+\dots} = e^{Y+\text{ad}(X)Y+\dots}. \quad (6.5)$$

It turns out that in a basis, the matrix elements of the adjoint representation are given by the structure constants f_{ij}^k : Given a set of generators, the matrix element $(\text{ad } T_i)_j^k$ is given by the coefficient of T_k in $\text{ad } T_i \cdot T_j$, so we have

$$(\text{ad } T_i)_j^k = \text{ad } T_i \cdot T_j \Big|_{T_k} = [T_i, T_j] \Big|_{T_k} = f_{ij}^k. \quad (6.6)$$

The matrices of the adjoint representation are purely real, since the f_{ij}^k are. This is consistent with anti-Hermiticity since the T_i are real and antisymmetric. In a convention with Hermitian generators, the matrix elements of the adjoint representation will be imaginary.

6.1.4 Killing Form

Since in a representation, the product is well-defined, one can define a kind of scalar product on the adjoint representation:

Definition 25. *The Killing form is a bilinear form on the Lie algebra defined via the adjoint representation as*

$$(X, Y) = k \text{tr ad}(X) \text{ad}(Y). \quad (6.7)$$

Here k is a normalisation constant.

The trace again ensures that equivalent representations have the same Killing form. Note that this scalar product is not necessarily positive definite!

If we have chosen a basis of generators T_i , we can associate a matrix g_{ij} to the Killing form as

$$g_{ij} = \text{tr ad}(T_i) \text{ad}(T_j) = f_{il}^k f_{jk}^l, \quad (6.8)$$

so it is also fixed by the structure constants. Clearly, g is a real symmetric matrix.

The “metric” g_{ij} contains a fair bit of information. In particular, we have that the group \bar{G} is compact if g_{ij} (or the Killing form) is negative definite, in which case also the algebra is called compact (although it clearly is non-compact as a space!). Note that the Killing form can never be positive definite.

If g_{ij} is nondegenerate, we can use it to raise and lower indices. In particular, we can define structure constants with only lower indices by

$$f_{ijk} = f_{ij}^l g_{lk} = \text{tr}([T_i, T_j] T_k). \quad (6.9)$$

They are completely antisymmetric because the right-hand side is. We will often take a “diagonal” form of the generators such that $g_{ij} = -\lambda \delta_{ij}$, with λ a positive constant. Then we can basically identify upper and lower indices (up to a factor of λ).

6.1.5 Subalgebras

A Lie algebra \mathfrak{g} might have *subalgebras*, i.e. vector subspaces $\mathfrak{h} \subset \mathfrak{g}$ which themselves form a Lie algebra (with the same Lie bracket). Subalgebras generate subgroups. A special type of subgroup is an *invariant subgroup*: \mathfrak{h} is an invariant subgroup if for every $H \in \mathfrak{h}$ and $X \in \mathfrak{g}$, the commutator $[H, X]$ is again in \mathfrak{h} . (In algebra language, \mathfrak{h} is an ideal.) Invariant subalgebras generate normal subgroups. In particular, the *center* of the algebra, which is the set of those elements that commute with everything,

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \forall Y \in \mathfrak{g}\} \quad (6.10)$$

is an ideal. An algebra that does not contain nontrivial ideals¹ is called *simple*. A simple algebra generates a *simple Lie group*, which means a Lie group which does not contain a continuous normal subgroup – discrete normal subgroups are still allowed, so a simple Lie group need not be a simple group (which has no normal subgroups).

An algebra that does not contain Abelian invariant subalgebras (i.e. for which the Lie bracket is identically zero) is called *semisimple*. In particular, for semisimple algebras, the center is trivial. Semisimple algebras are direct sums of simple algebras.

The semisimplicity of an algebra can again be analysed by looking at the Killing form: The algebra is semisimple if and only if the Killing form is nondegenerate.

We prove half of this theorem: For the first half, we show that if \mathfrak{g} is not semisimple, i.e. it has an Abelian ideal B , the Killing form is degenerate. Choose a basis T_i of \mathfrak{g} such that the first m elements span B and the remaining $T_i \notin B$. Then the Killing matrix elements g_{1i} are

$$g_{1i} = f_{1l}^j f_{ij}^l = \sum_{k=l} f_{1k}^j f_{ij}^l. \quad (6.11)$$

The summand is obtained by

$$f_{1k}^j f_{ij}^l T_l = [T_i, [T_1, T_k]]. \quad (6.12)$$

We can consider three cases:

- If $T_k \in B$, the inner commutator vanishes because B is Abelian.
- If $T_i \in B$, the outer commutator vanishes because the inner commutator is in B .
- If both $T_i, T_k \notin B$, the inner and thus the outer commutator are in B , hence $T_l \in B$, and so the summand vanishes for $k = l$.

Hence $g_{1i} = 0$, and the Killing form is degenerate.

The converse is somewhat harder to show, so we just give a sketch: The key point is that the null space of the Killing form, i.e. those X for which $(X, Y) = 0$ for all Y , is an ideal. We can bring the matrices to Jordan form, and then we see that the X in the null

¹Clearly, 0 and \mathfrak{g} itself are trivially ideals, so nontrivial means other than those.

space must have eigenvalues equal to zero, i.e. they are purely nilpotent. Then some commutator power of them will be a nonzero Abelian ideal, hence \mathfrak{g} is not semisimple.

In a semisimple algebra, every generator thus has a nonzero commutator with at least one other element. Furthermore, since the Killing form is nondegenerate, we can bring it to a diagonal form with all entries ± 1 . In other words, the structure constants can be “inverted” in the sense that every generator can be expressed as a linear combination of commutators, hence it is in particular traceless.

Finally, note that the adjoint representation is irreducible for a simple algebra. If the algebra is semisimple, the adjoint representation is reducible: The ideals form invariant subspaces. If \mathfrak{g} contains Abelian ideals, the adjoint representation is not faithful, while for semisimple algebras it is. Furthermore, for semisimple algebras, every representation is completely reducible. Since a semisimple Lie algebra is a direct sum of simple ones, so it suffices to consider simple Lie algebras in the following.

6.1.6 Real and Complex Lie Algebras

Up to now, we have (implicitly) restricted ourselves to real Lie algebras (i.e. algebras over the real numbers), as those are what you get from a real manifold, since for complex algebras it does not make much sense to choose Hermitean or anti-Hermitean generators, and the Killing form is never definite. Note that a real algebra can involve complex numbers – for example, $\mathfrak{su}(n)$ contains complex matrices, but it is a real algebra, since otherwise, restricting to (anti-)Hermitean matrices would not make sense.

However, in the following it will be useful to consider complex algebras, as we will see. Hence we will at some point pass to the complexified algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ (and we usually omit the subscript). In practice, this means we retain the basis of generators, but allow for complex coefficients. We will also consider representations on complex vector spaces.

As an aside, note that for a given complex Lie algebra, there are several real algebras of which it is the complexification. In particular, there are two distinguished real forms: The *compact real form* for which the Killing form is negative definite and which consequently generates a compact group, and the *normal real form*, for which the Killing form is maximally non-definite, i.e. for which the number of positive and negative eigenvalues of the Killing form is equal (or different by one, if the dimension is odd). We already discussed the examples $\mathfrak{so}(2)$ and $\mathfrak{so}(1,1)$ at the beginning of the previous chapter – they are the two real forms of one complex algebra (basically two ways of embedding \mathbb{R} in \mathbb{C}). We will not go into more detail here and consider the complex form most of the time, but with the compact real form in mind.

6.2 Representations of $\mathfrak{su}(2)$

We will first discuss representations of $\mathfrak{su}(2)$ (which you probably already know), to obtain some intuition. In the next Section we will then generalise this to arbitrary

simple algebras. We will switch conventions here: In the real form of the algebras, we consider Hermitean generators and real structure constants.

6.2.1 Diagonalising the Adjoint Representation

The algebra $\mathfrak{su}(2)$ contains three Hermitean generators J_i with commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \quad (6.13)$$

This means that in the adjoint representation no generator is diagonal, e.g. $\text{ad}(J_1) \cdot J_2 = iJ_3$. As a first step, we will diagonalise one generator (since they do not commute, we cannot diagonalise more than one), and it is customary to take J_3 . An easy calculation shows that suitable linear combinations are

$$J_{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2). \quad (6.14)$$

Note that we have to allow complex coefficients here. The algebra is thus no longer compact, as the J_{\pm} are not Hermitean, but satisfy $J_{\pm}^{\dagger} = J_{\mp}$. However, we gain a diagonal action of J_3 :

$$[J_3, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = J_3. \quad (6.15)$$

6.2.2 Constructing an Irreducible Representation

Now let us consider a (as always finite-dimensional) representation on a complex vector space. Since we have diagonalised J_3 in the adjoint representation, we will also choose a basis of our representation space where J_3 is diagonal (remember J_3 is still Hermitean, so this is always possible, and the eigenvalues are real.) Denote the largest eigenvalue by j ,

$$J_3 |j, \alpha\rangle = j |j, \alpha\rangle. \quad (6.16)$$

Here α denotes possible other quantum numbers characterising the state. We choose a normalisation

$$\langle j, \alpha | j, \beta \rangle = \delta_{\alpha\beta}. \quad (6.17)$$

The commutation relations tell us that J_{\pm} raise or lower the J_3 eigenvalue by one,

$$J_3 J_{\pm} |j, \alpha\rangle = (J_{\pm} J_3 + [J_3, J_{\pm}]) |j, \alpha\rangle = (j \pm 1) J_{\pm} |j, \alpha\rangle. \quad (6.18)$$

Hence they called *raising* and *lowering operators*.

Since we have assumed that j is the largest eigenvalue of J_3 , we see that $J_+ |j, \alpha\rangle = 0$. On the other hand, J_- will create a state with eigenvalue $j - 1$,

$$J_- |j, \alpha\rangle = N_j |j - 1, \alpha\rangle. \quad (6.19)$$

For the normalisation, observe first that this state is orthogonal to $|j, \alpha\rangle$,

$$\langle j, \alpha | j - 1, \alpha \rangle = \frac{1}{N_j} \langle j, \alpha | J_- | j, \alpha \rangle = 0, \quad (6.20)$$

since $\langle j, \alpha | J_- = (J_+ | j, \alpha \rangle)^\dagger$. You can convince yourself that this generalises to $\langle j - k, \alpha | j - l, \alpha \rangle \sim \delta_{kl}$. Furthermore, we see that the orthogonality for different α persists,

$$\begin{aligned} N_j^* N_j \langle j - 1, \alpha | j - 1, \beta \rangle &= \langle j, \alpha | J_+ J_- | j, \beta \rangle \\ &= \langle j, \alpha | [J_+, J_-] | j, \beta \rangle = j \langle j, \alpha | j, \beta \rangle = j \delta_{\alpha\beta}. \end{aligned} \quad (6.21)$$

Hence we can choose a real normalisation, $N_j = \sqrt{j}$ (this involves a choice of the phase of what we call $|j - 1, \alpha\rangle$). We can raise the J_3 eigenvalue again by applying J_+ ,

$$\begin{aligned} J_+ |j - 1, \alpha\rangle &= \tilde{N}_j |j, \alpha\rangle \\ &= \frac{1}{N_j} J_+ J_- |j, \alpha\rangle = \frac{1}{N_j} [J_+, J_-] |j, \alpha\rangle = \frac{j}{N_j} |j, \alpha\rangle = \sqrt{j} |j, \alpha\rangle. \end{aligned} \quad (6.22)$$

So we see that the normalisation constants are equal, $\tilde{N}_j = N_j$.

Clearly now, we can generate more states by acting repeatedly with J_- , and we go back by applying J_+ . A simple generalisation of the above argument shows that all these states will be orthogonal in α -space. Hence we drop the extra label henceforth, since each value of α will simply lead to another copy of the same representation. (We have already implicitly assumed that N_j does not depend on α). All that remains is to find a general expression for the normalisations. To this end, we take

$$J_- |j - k\rangle = N_{j-k} |j - k - 1\rangle, \quad J_+ |j - k - 1\rangle = N_{j-k} |j - k\rangle. \quad (6.23)$$

By the same reasoning as above, the numbers are equal and can be chosen real.

Imposing the algebra, we find a recursion relation,

$$\begin{aligned} N_{j-k}^2 &= \langle j - k | J_+ J_- | j - k \rangle = \langle j - k | [J_+, J_-] | j - k \rangle + \langle j - k | J_- J_+ | j - k \rangle \\ &= j - k + N_{j-k+1}^2. \end{aligned} \quad (6.24)$$

With the boundary condition $N_j = \sqrt{j}$, we find the solution

$$N_{j-k} = \frac{1}{\sqrt{2}} \sqrt{(2j - k)(k + 1)} = \frac{1}{\sqrt{2}} \sqrt{(j + m)(j - m + 1)}, \quad (6.25)$$

where we have introduced the usual notation $j - k = m$.

In principle we could continue this indefinitely – but since we want finite-dimensional representations, the series should terminate at some smallest m , which can only be $m = -j$. Hence, m takes values from j to $-j$, with integer steps. From this it follows that $2j$ must be an integer, and m takes $2j + 1$ values. Indeed, the trace of any operator

is zero: J_{\pm} are purely off-diagonal, and the eigenvalues of J_3 are symmetric around zero. (This is easy to see in this basis, but the trace is invariant under changes of basis.)

In summary, any irreducible representation of $\mathfrak{su}(2)$ is labelled by a nonnegative integer or half-integer j , and this representation is $2j + 1$ dimensional. The states within the representation j are labelled by the integer or half-integer J_3 eigenvalue m , ranging from j to $-j$.

In quantum mechanics, j is sometimes called the spin of the representation, and m is the “ z -component”. In general group theory terms, the J_3 eigenvalue is the *weight* of the state. j is called the highest weight of the representation, and this method is the *highest weight construction*.

The representations with integral and half-integral j differ in an important respect: You saw on the problem sheet that $SU(2)$ elements are given by an expression of the form

$$U(\hat{n}, \phi) = e^{i\phi\hat{n}\cdot\vec{J}} \quad (6.26)$$

with ϕ being 4π -periodic, $U(\hat{n}, 4\pi) = U(\hat{n}, 0) = \mathbb{1}$, while elements of $SO(3)$ are similar, but with periodicity 2π . For the representations we just found, J_3 is diagonal, with entries $j, j - 1, \dots, -j$, so we see that

$$e^{i\phi J_3} = \begin{pmatrix} e^{ij\phi} & & & \\ & e^{i(j-1)\phi} & & \\ & & \ddots & \\ & & & e^{-ij\phi} \end{pmatrix}. \quad (6.27)$$

Hence it has periodicity 2π for integral and 4π for half-integral j – that means that only integral j ’s lead to representations of $SO(3)$! This is the group-theoretic basis for the fact that orbital angular momentum (which is related to actual rotations in space, hence $SO(3)$) has integer l , while the spin can take values in representations of the universal covering group of $SO(3)$, which is $SU(2)$.

Note also that the adjoint representation corresponds to $j = 1$. This is already apparent since it is three-dimensional. It can be explicitly constructed starting from the highest weight state $|j = 1, m = 1\rangle = J_+$: The states are

$$\begin{aligned} |j = 1, m = 1\rangle &= J_+, \\ |j = 1, m = 0\rangle &= \text{ad } J_- J_+ = -J_3, \\ |j = 1, m = -1\rangle &= \text{ad } J_- (-J_3) = -J_-. \end{aligned} \quad (6.28)$$

6.2.3 Decomposing a General Representation

Finally, note that the preceding procedure does not necessarily exhaust the full representation space: Not every state with weight $m < j$ must descend from a weight- j

state via lowering operators. Rather, the representation space might be a direct sum of irreducible representations of different highest weight,

$$V = \bigoplus_{j=0}^{j_{\max}} V_j^{\oplus n_j}. \quad (6.29)$$

Each representation can appear with a multiplicity n_j – this is what is counted by the label α . To disentangle this, we now label the states with the J_3 eigenvalue m and the highest weight j . One can show that states with different j are orthogonal,

$$\langle j, m | j', m' \rangle \sim \delta_{jj'} \delta_{mm'}, \quad (6.30)$$

so the procedure to reduce a given representation into its irreducible components is as follows:

1. Diagonalise J_3 – its eigenvalues must be integers or half-integers (for reducible representations, both can appear!), symmetrically arranged around zero.
2. Start with the highest eigenvalue j_{\max} and go through the procedure outlined above. You have now identified n_j copies of representation j_{\max} , $V_{j_{\max}}^{\oplus n_j}$.
3. Take the orthogonal complement of this subspace and repeat the procedure.
4. After a finite number of steps, you have decomposed the complete representation space.

6.3 The Cartan–Weyl Basis

Now we extend the $\mathfrak{su}(2)$ discussion to arbitrary semisimple compact Lie algebras. This brings a separation of the algebra into two pieces: the Cartan subalgebra, which is the analogue of J_3 , and the roots, which correspond to the raising and lowering operators. Note that, in contrast to many other cases in physics, here it is convenient to single out a particular (type of) basis instead of working basis-independent.

6.3.1 The Cartan Subalgebra

The first thing to do is again to “diagonalise the adjoint representation”, i.e. diagonalise the maximal number of elements. To that end, we need to find a maximal set of Hermitean elements H_i that commute among themselves,

$$[H_i, H_j] = 0, \quad H_i^\dagger = H_i. \quad (6.31)$$

Definition 26. *The H_i are called the Cartan generators, and they span the Cartan subalgebra \mathfrak{h} . The number of Cartan generators (i.e the dimension of \mathfrak{h}) is called the rank r of the Lie algebra \mathfrak{g} .*

Recall that two commuting elements also commute as operators in the adjoint representation. This is the analogue of J_3 in the $\mathfrak{su}(2)$ case. There is no element that commutes with J_3 , so $\mathfrak{su}(2)$ has rank one.

Note that the Cartan algebra is not unique - indeed in our $\mathfrak{su}(2)$ example we chose $H = J_3$, but we could just as well have chosen J_1 . However, different choices of \mathfrak{h} are related by automorphisms of \mathfrak{g} , so they do not lead to different results. In particular, any Cartan subalgebra has the same dimension, so the rank is well-defined. (The construction of such an automorphism is described in the book by Fuchs and Schweigert, Section 11.3.)

Since we will deal with the complexification of the algebra, it is convenient at this point to define a new scalar product that involves Hermitean conjugation of the first factor:

$$\langle A, B \rangle = k \operatorname{tr} A^\dagger B. \quad (6.32)$$

This product is always positive definite. The normalisation constant k is chosen equal to the k in the Killing form (6.7), so the two scalar products coincide for Hermitean operators. Hence, we can choose the Cartan generators to be orthogonal, $\langle H_i, H_j \rangle = \delta_{ij}$, and they are still orthogonal in the Killing form. What is more, Hermitean matrices are self-adjoint with respect to the product $\langle \cdot, \cdot \rangle$ in the sense that

$$\begin{aligned} \langle A, \operatorname{ad} H \cdot B \rangle &= \langle A, [H, B] \rangle = k \operatorname{tr} A^\dagger [H, B] = k \operatorname{tr} [A^\dagger, H] B = k \operatorname{tr} [H^\dagger, A]^\dagger B \\ &= \langle \operatorname{ad} H^\dagger \cdot A, B \rangle, \end{aligned} \quad (6.33)$$

i.e. an operator is self-adjoint if it is Hermitean as a matrix.

6.3.2 Roots

After we have fixed the Cartan subalgebra, we go on diagonalising the elements. This means we choose a basis E_α of the remaining algebra satisfying

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \quad (6.34)$$

This usually again requires that we pass to the complexified Lie algebra, i.e. we allow complex coefficients. The eigenvalues α_i are still real (since the H_i are Hermitean) and are called roots. Each non-Cartan element is labelled by an r -component root vector $\alpha = (\alpha_1, \dots, \alpha_r)$, and we will later see that α determines the generator uniquely. (Sometimes the E_α generators are called roots themselves.) The set of all roots is called the root space. The roots cannot be Hermitean, because we have

$$(\alpha_i E_\alpha)^\dagger = \alpha_i E_\alpha^\dagger = [H_i, E_\alpha]^\dagger = -[H_i, E_\alpha^\dagger], \quad (6.35)$$

so actually we see that $E_\alpha^\dagger = E_{-\alpha}$. This is the analogue of $J_\pm^\dagger = J_\mp$ in $\mathfrak{su}(2)$. In particular, this means that if α is a root, then $-\alpha$ is as well.

Since the E_α with different roots have different eigenvalues with the Hermitean Cartan generators, they are orthogonal, $\langle E_\alpha, E_\beta \rangle = 0$ for $\alpha \neq \beta$. We will choose them orthonormal,

$$\langle E_\alpha, E_\beta \rangle = k \operatorname{tr} E_\alpha^\dagger E_\beta = k \operatorname{tr} E_{-\alpha} E_\beta = \delta_{\alpha\beta}. \quad (6.36)$$

To find the commutator of the roots, observe that

$$[H_i, [E_\alpha, E_\beta]] = (\alpha_i + \beta_i) [E_\alpha, E_\beta]. \quad (6.37)$$

Hence, in particular, $[E_\alpha, E_{-\alpha}]$ commutes with all the Cartan generators. Since the Cartan subalgebra is maximal, it must be a linear combination,

$$[E_\alpha, E_{-\alpha}] = \sum_i c_i H_i. \quad (6.38)$$

To find the coefficient, we take the scalar product with H_j and exploit the cyclicity of the trace,

$$c_j = \langle [E_\alpha, E_{-\alpha}], H_j \rangle = k \operatorname{tr} [H_j, E_\alpha] E_{-\alpha} = \alpha_j \langle E_\alpha, E_\alpha \rangle = \alpha_j. \quad (6.39)$$

Hence,

$$[E_\alpha, E_{-\alpha}] = \sum_i \alpha_i H_i = \vec{\alpha} \cdot \vec{H}. \quad (6.40)$$

Before we go on and discuss commutators of different roots, we first exploit a nice property of a root pair $E_{\pm\alpha}$: It forms a $\mathfrak{su}(2)$ subalgebra. Specifically, the properly normalised subalgebra is spanned by

$$\begin{aligned} E_\pm &= \frac{1}{|\alpha|} E_{\pm\alpha}, \\ E_3 &= \frac{1}{|\alpha|^2} \vec{\alpha} \cdot \vec{H}. \end{aligned} \quad (6.41)$$

Here $|\alpha|^2 = \vec{\alpha} \cdot \vec{\alpha}$ is the usual Euclidean norm of the vector. We already know everything about the representations of $\mathfrak{su}(2)$, and since this subalgebra acts on all of \mathfrak{g} via the commutator, \mathfrak{g} must decompose into irreducible representations of $\mathfrak{su}(2)$. This will now help us to establish two facts:

1. The root vector uniquely determines a generator.
2. The only multiples of a root which are roots themselves are $\pm\alpha$.

To show the first statement, assume the contrary: Take E_α and E'_α be two generators with the same root vector. We can choose them to be orthogonal,

$$\langle E_\alpha, E'_\alpha \rangle = k \operatorname{tr} E_{-\alpha} E'_\alpha = 0. \quad (6.42)$$

Now E'_α must be a state in some spin- j representation of the $\mathfrak{su}(2)$ generated by E_α , Eq. (6.41). We can act on it with the lowering operator E_- , and we find that by Eq. (6.37), the commutator $[E_-, E'_\alpha] \sim [E_{-\alpha}, E'_\alpha]$ commutes with all the H_i . Thus, it is a linear combination of Cartan generators, $[E_{-\alpha}, E'_\alpha] = c_i H_i$. However, we find that

$$c_i = \langle H, [E_{-\alpha}, E'_\alpha] \rangle = k \operatorname{tr} E_{-\alpha} [E'_\alpha, H_i] = -\alpha_i k \operatorname{tr} E_{-\alpha} E'_\alpha = 0. \quad (6.43)$$

Hence, $[E_-, E'_\alpha] = 0$ – the lowering operator annihilates the state E'_α , hence it must be the lowest weight state of an $\mathfrak{su}(2)$ representation! In particular, the E_3 eigenvalue must be $-j$. On the other hand, a direct calculation shows that

$$[E_3, E'_\alpha] = \frac{1}{|\alpha|^2} \vec{\alpha} \cdot [\vec{H}, E'_\alpha] = E'_\alpha, \quad (6.44)$$

so its weight is one. So we find a contradiction: The lowest weight of an $\mathfrak{su}(2)$ representation cannot be positive. Hence, there is no such E'_α , and each α fixes a unique generator.

For the second statement, assume there is a generator $E_{\lambda\alpha}$ with some $\lambda \neq \pm 1$. Then this has E_3 eigenvalue λ ,

$$[E_3, E_{\lambda\alpha}] = \frac{1}{|\alpha|^2} \vec{\alpha} \cdot [\vec{H}, E_{\lambda\alpha}] = \lambda E_{\lambda\alpha}. \quad (6.45)$$

Hence λ must be an integer or half-integer. Assume first that λ is an integer. Then we can apply E_+ or E_- a number of times and arrive at a state with eigenvalue one. This, however, cannot be E_α , since that one is (as E_+) part of a complete spin-one representation. Hence, it is another state with the same root vector, which does not exist. If now λ is a half-integer, we can redefine

$$E_{\lambda\alpha} \longrightarrow E_\beta, \quad E_\alpha \longrightarrow E_{\beta/\lambda} \quad (6.46)$$

and apply the argument for β . Hence, only α and $-\alpha$ are roots.

Finally, what can we actually say about the commutators of different roots? Not so much in general, but there is something. Recall that $[E_\alpha, E_\beta]$ has root vector $\alpha + \beta$. The case $\alpha = -\beta$ was already discussed, see Eq. (6.40). Now consider the case $\alpha + \beta \neq 0$. Then we have to distinguish whether $\alpha + \beta$ is a root. If it is, we know that it is unique, so we can conclude that the commutator is proportional to that generator,

$$[E_\alpha, E_\beta] = e_{\alpha,\beta} E_{\alpha+\beta} \quad \text{for } \alpha + \beta \text{ a root.} \quad (6.47)$$

$e_{\alpha,\beta}$ is generically non-zero. It is constrained by the Jacobi identity: One can deduce that

$$e_{\alpha,\beta} = e_{\beta,-(\alpha+\beta)} = e_{-(\alpha+\beta),\alpha}. \quad (6.48)$$

On the other hand, if $\alpha + \beta$ is not a root,

$$[E_\alpha, E_\beta] = 0 \quad \text{for } \alpha + \beta \text{ not a root.} \quad (6.49)$$

In summary, the algebra decomposes into the Cartan subalgebra, which is a maximal set of mutually commuting elements, and the root generators, each of which is labelled by an r -component root vector α . The commutation relations are

$$\begin{aligned} [H_i, H_j] &= 0, & [H_i, E_\alpha] &= \alpha_i E_\alpha, \\ [E_\alpha, E_\beta] &= \begin{cases} \vec{\alpha} \cdot \vec{H} & \text{if } \alpha = -\beta \\ e_{\alpha, \beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root} \\ 0 & \text{if } \alpha + \beta \text{ is not a root} \end{cases} \end{aligned} \quad (6.50)$$

The basis of the Lie algebra consisting of the H_i and E_α is called a Cartan–Weyl basis. Note that this basis is not unique: We have to choose a Cartan subalgebra, and its basis, which, even after imposing orthonormality, is still subject to orthogonal transformations. Furthermore, we have to choose particular root elements, which, however, involve only a normalisation freedom. There is a further restricted basis, called the Chevalley–Serre basis, but we will not go into this (see the book by Fuchs and Schweigert for details).

6.4 Roots and Weights

Now that we have found a nice basis for the algebra, i.e. the adjoint representation, let us consider a general irreducible representation ρ . We can again diagonalise the Cartan elements, such that the states obey

$$H_i |\omega\rangle = \omega_i |\omega\rangle. \quad (6.51)$$

The r -component vector ω is called the *weight vector* (or just weight) of the state, and the number of distinct weight vectors is the dimension of the representation. The set of all weights is called the weight space. In the adjoint representation, we have $\omega_{\text{ad}} = \alpha$ – the roots are the weights of the adjoint representation! Then we immediately see that the E_α are the analogues of J_\pm , i.e. they change the eigenvalue of a state:

$$H_i E_\alpha |\omega\rangle = (\omega_i + \alpha_i) E_\alpha |\omega\rangle. \quad (6.52)$$

It is not yet defined whether E_α is a raising or lowering operator – later we will choose a convention for that. So either $\omega + \alpha$ is again a weight of the representation, then we have

$$E_\alpha |\omega\rangle = N_{\alpha, \omega} |\omega + \alpha\rangle, \quad (6.53)$$

or $\omega + \alpha$ is not a weight, then the state is annihilated, $E_\alpha |\omega\rangle = 0$. Note that this also implies that the difference of two weights of an irreducible representation is a sum of roots.

6.4.1 The Master Formula

Now we can, for a given root vector, again exploit the $\mathfrak{su}(2)$ subalgebra (6.41). Under this $\mathfrak{su}(2)$, each state must transform in an irreducible spin- j representation. For a

given ω , we don't know the m value, but we know that we can find all the states in the representation by applying the E_{\pm} operators a finite number of times. In other words, there are nonnegative integers p and q such that the representation contains the states

$$E_-^q |\omega\rangle, E_-^{q-1} |\omega\rangle, \dots, E_- |\omega\rangle, |\omega\rangle, E_+ |\omega\rangle, \dots, E_+^p |\omega\rangle, \quad (6.54)$$

but no more,

$$E_-^{q+1} |\omega\rangle = 0 = E_+^{p+1} |\omega\rangle. \quad (6.55)$$

The spin of this representation clearly is $j = \frac{1}{2}(p + q)$.

The weights of the states range from $\omega - q\alpha$ to $\omega + p\alpha$. Their E_3 eigenvalues are

$$E_3 |\omega + n\alpha\rangle = \frac{1}{|\alpha|^2} \vec{\alpha} \vec{H} |\omega + n\alpha\rangle = \frac{(\vec{\omega} + n\vec{\alpha}) \cdot \vec{\alpha}}{|\alpha|^2} |\omega + n\alpha\rangle. \quad (6.56)$$

They must range from $-j$ to j ,

$$-\left(\frac{\vec{\omega} \cdot \vec{\alpha}}{|\alpha|^2} - q\right) = j = \frac{\vec{\omega} \cdot \vec{\alpha}}{|\alpha|^2} + p, \quad (6.57)$$

from which we again see that $j = \frac{1}{2}(p + q)$, and

$$\frac{\vec{\omega} \cdot \vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}} = -\frac{1}{2}(p - q). \quad (6.58)$$

This formula looks innocent, but will be very important – we will refer to this as the master formula. Part of its power is that it works in both directions, that is, if there is a vector ω that satisfies the master formula for all roots and nonnegative integers p and q , then ω is a weight vector.

As a side remark, the fact that the E_3 eigenvalues are symmetric around zero leads to a symmetry operation of the weights, the so-called Weyl reflection: The weight space is mapped to itself under

$$\omega \longmapsto \omega - 2 \frac{\vec{\omega} \cdot \vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}} \alpha, \quad p \longmapsto q, \quad q \longmapsto p. \quad (6.59)$$

Geometrically, this corresponds to a reflection of the weights through the hyperplane perpendicular to the root α . In the adjoint representation, Weyl reflections form part of the automorphism group of the Lie algebra.

6.4.2 Geometry of Roots

As mentioned, roots are the weights of the adjoint representation, so we can apply the master formula both ways. Take two roots α and β . Then we get

$$\frac{\vec{\beta} \cdot \vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}} = -\frac{1}{2}(p - q) \quad \text{and} \quad \frac{\vec{\alpha} \cdot \vec{\beta}}{\vec{\beta} \cdot \vec{\beta}} = -\frac{1}{2}(p' - q'). \quad (6.60)$$

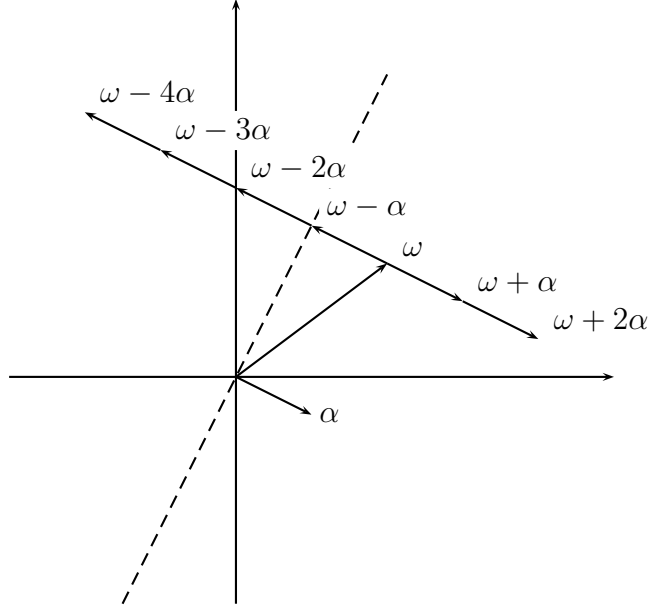


Figure 6.1: The weight ω is part of a $j = 3$ representation of the $\mathfrak{su}(2)_\alpha$ in the weight space. Weyl reflections are reflections at the dashed line orthogonal to the root α .

Taking the product of both formulae, we obtain

$$\frac{(\vec{\beta} \cdot \vec{\alpha})^2}{\vec{\alpha}^2 \vec{\beta}^2} = \cos^2 \theta = \frac{1}{4} (p - q) (p' - q') = \frac{n}{4}. \quad (6.61)$$

Here θ is the angle between the roots, and n is an integer. Since the cosine is bounded, n can take only take values from zero to four. Actually, four is not very interesting – it corresponds to $\theta = 0$, which means $\alpha = \beta$ (because of uniqueness), or $\theta = \pi = 180^\circ$, i.e. $\alpha = -\beta$, which is always there. Hence there are only four nontrivial angles.

Taking the quotient of the original equations, we can also get some information on the possible lengths of α and β , or actually their ratio,

$$\frac{\vec{\alpha}^2}{\vec{\beta}^2} = \frac{p' - q'}{p - q}. \quad (6.62)$$

Again, given that p, q, p' and q' are integers and their product must be zero to three, we have only the four possibilities summarised in the table:

Hence the root diagram is rather constrained.

6.4.3 Example: $\mathfrak{su}(3)$

After all the dry theory, let us consider an example: $\mathfrak{su}(3)$. Since $SU(3)$ consists of the unitary matrices with determinant one, $\mathfrak{su}(3)$ contains the traceless Hermitean matrices

n	θ	$\frac{\alpha^2}{\beta^2}$
0	90°	arbitrary
1	60°, 120°	1
2	45°, 135°	$\frac{1}{2}$, 2
3	30°, 150°	$\frac{1}{3}$, 3

Table 6.1: The possible angles and relative lengths of roots.

(in physicist's convention), which is an eight-dimensional space. The customary basis is $T_a = \lambda_a/2$, where the Gell-Mann matrices λ_a are

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{6.63}$$

The first three are an obvious embedding of the Pauli matrices of $\mathfrak{su}(2)$. They are normalised to $\text{tr } T_a T_b = \frac{1}{2} \delta_{ab}$. This is chosen such that $[T_1, T_2] = iT_3$. To make it consistent with $\langle T_i, T_j \rangle = \delta_{ij}$, we choose the normalisation constant in (6.7) and (6.32) to be $k = 2$. Then the Killing metric is $g_{ij} = \delta_{ij}$, and we do not have to care about upper and lower indices on the structure constants, i.e. $f_{abc} = f_{ab}^c$. The independent nonvanishing structure constants are

$$\begin{aligned}
f^{123} &= 2 f^{147} = 2 f^{246} = 2 f^{257} = 2 f^{345} = -2 f^{156} \\
&= -2 f^{367} = \frac{2}{\sqrt{3}} f^{458} = \frac{2}{\sqrt{3}} f^{678} = 1.
\end{aligned} \tag{6.64}$$

This algebra has rank two. As Cartan generators one usually chooses $H_1 = T_3$ and $H_2 = T_8$, which are already diagonal, so they commute. To find the roots, we have to diagonalise the adjoint action of the Cartan elements. A straightforward calculation gives

$$\begin{aligned}
E_{\pm(1,0)} &= \frac{1}{\sqrt{2}} (T^1 \pm iT^2), & E_{\pm(\frac{1}{2}, \frac{\sqrt{3}}{2})} &= \frac{1}{\sqrt{2}} (T^4 \pm iT^5), \\
E_{\pm(-\frac{1}{2}, \frac{\sqrt{3}}{2})} &= \frac{1}{\sqrt{2}} (T^6 \pm iT^7).
\end{aligned} \tag{6.65}$$

So the roots are

$$\alpha^1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \alpha^2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \quad \alpha^3 = (1, 0) \quad (6.66)$$

and their negatives. The roots are shown in Figure 6.2. We will use a notation where superscripts label the roots, while subscripts label the vector components. (So the subscripts are actually lower indices, while the superscripts are not proper indices, and they do not take part in the summation convention. This is even more confusing because both sub- and superscripts have the same range.)

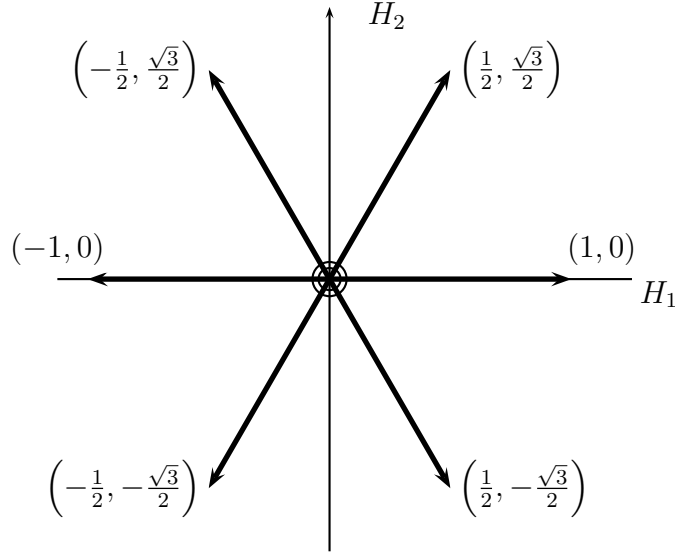


Figure 6.2: The roots of $\mathfrak{su}(3)$. Note that indeed the angles are multiples of 60° . The circles around the origin denote the two Cartan generators which have “root vector” zero.

Of course, the T_a not only give the adjoint representation by acting on themselves, but they naturally act on \mathbb{C}^3 . For any matrix algebra, this is called the *defining* or *vector representation*, and it is denoted by its dimension as $\mathbf{3}$. Since they are already diagonal, the eigenvalues of H_1 and H_2 are simply the diagonal elements, and the eigenvectors are the standard basis of \mathbb{C}^3 . Hence the weights are

$$\begin{aligned} |\omega^1\rangle &\equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left| \left(\frac{1}{2}, \frac{1}{2\sqrt{3}} \right) \right\rangle, & |\omega^2\rangle &\equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \left| \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}} \right) \right\rangle, \\ |\omega^3\rangle &\equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \left| \left(0, -\frac{1}{\sqrt{3}} \right) \right\rangle. \end{aligned} \quad (6.67)$$

Note that indeed the differences of weights are roots.

There is even a third representation we can construct from the T_a 's, which is called the *complex conjugate representation*: Clearly, if the generators T_a form an algebra with structure constants f_{ab}^c , then so do the generators $-T_a^*$. This is of course true for any representation. Since the Cartan generators are diagonal and real, the weights just receive an overall minus sign, in particular, they are different. (This is in contrast to the adjoint representation, which is isomorphic to its complex conjugate representation.) So we have a representation, again three-dimensional, called $\bar{\mathbf{3}}$, with states

$$\langle \nu^1 | = \left\langle -\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \right|, \quad \langle \nu^2 | = \left\langle -\left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \right|, \quad \langle \nu^3 | = \left\langle \left(0, \frac{1}{\sqrt{3}}\right) \right|. \quad (6.68)$$

The weights of the vector representation and its conjugate are depicted in Figure 6.3.

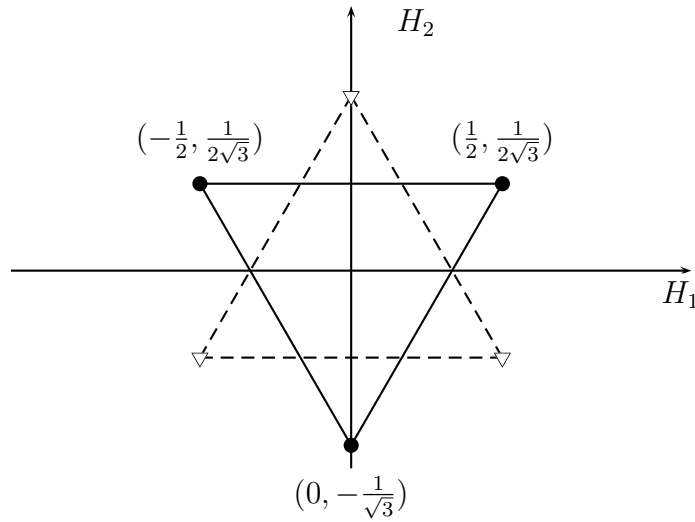


Figure 6.3: The weights of the $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $\mathfrak{su}(3)$. The lines connecting the weights are roots.

6.5 Positive and Simple Roots

For $\mathfrak{su}(2)$ we had a notion of raising and lowering operators, and of highest weight. To generalise that, we have to introduce an order on the space of weights. Clearly this will not be unique, but luckily, the results will not depend on the particular ordering.

The simplest way to define an order is the following: Fix a basis H_i of Cartan generators. In this basis, we define a weight vector ω to be positive, $\omega > 0$, if the first nonzero component is positive. Then an obvious order on weight space is given by

$$\omega > \mu \quad \Leftrightarrow \quad \omega - \mu \text{ is positive} . \quad (6.69)$$

Thus we now know what the highest weight of a representation is.

This also applies to the roots, so we have identified the raising operators: E_α is a raising operator if $\alpha > 0$, and a lowering operator otherwise. For $\mathfrak{su}(3)$, the positive roots are those on the right-hand side of diagram 6.2.

We can restrict our discussion to positive roots, because the other ones are just their negatives. But actually, we can restrict even more: Clearly, roots can be expressed as sums of other roots. This prompts a choice of “elementary” ones:

Definition 27. *A simple root is a positive root that cannot be written as a sum of positive roots.*

In other words, the simple roots are the “smallest” of the positive roots.

Simple roots have a number of important properties. In particular, they already fix the algebra in the sense, that one can reconstruct all roots, and even the generators, from the simple roots. We will not go into much detail about the reconstruction, but we mention some properties that will be important.

- First, let α and β be different simple roots. Then $\alpha - \beta$ is not a root: $\alpha - \beta$ is not zero, since the roots are different. Say it is positive, then we have $\alpha = \alpha - \beta + \beta$, i.e. α can be written as a sum of positive roots. But α is simple, so $\alpha - \beta$ cannot be a root.
- Second, we can apply the master formula again,

$$\frac{\vec{\beta} \cdot \vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}} = -\frac{1}{2}(p - q) . \quad (6.70)$$

q measures how often we can subtract α from β without leaving root space. But we saw that already $\beta - \alpha$ is not a root, so $q = 0$ and

$$\vec{\beta} \cdot \vec{\alpha} = -\frac{1}{2}p \vec{\alpha} \cdot \vec{\alpha} \leq 0 . \quad (6.71)$$

By the same argument,

$$\vec{\alpha} \cdot \vec{\beta} = -\frac{1}{2}p' \vec{\beta} \cdot \vec{\beta} \leq 0 . \quad (6.72)$$

Hence the angle between simple roots and the relative lengths are

$$\cos \theta = -\frac{\sqrt{pp'}}{2} , \quad \frac{\alpha^2}{\beta^2} = \frac{p'}{p} . \quad (6.73)$$

In particular, the angle is constrained to be $90^\circ \leq \theta < 180^\circ$. The first constraint comes because the cosine is nonpositive, the second because the roots are positive, so they lie in a half-space.

- Third, simple roots are linearly independent. To see this, consider a linear combination

$$\gamma = \sum_{\alpha} c_{\alpha} \alpha, \quad (6.74)$$

And check whether we can find coefficients c_{α} such that $\gamma = 0$. Since all α are positive, the c_{α} cannot all have the same sign. Hence we can split γ into strictly positive and negative pieces,

$$\gamma = \sum_{c_{\alpha} > 0} c_{\alpha} \alpha - \left(- \sum_{c_{\alpha} < 0} c_{\alpha} \alpha \right) = \mu - \nu. \quad (6.75)$$

Now consider the norm of γ :

$$\gamma^2 = (\mu - \nu)^2 = \mu^2 + \nu^2 - 2\mu \cdot \nu. \quad (6.76)$$

Clearly, μ and ν cannot vanish, so their norm is positive. However, since μ and ν are both positive linear combinations of simple roots, their scalar product is negative, as we have seen above. Hence, the norm of γ never vanishes, so no linear combination of simple roots can be zero.

- What is more, the simple roots form a basis: If this was not the case, there would be a vector $\vec{\xi}$ which is orthogonal to all simple roots. But it is easy to see that any positive root can be written as a linear combination of simple roots with non-negative integer coefficients,

$$\gamma = \sum_{\alpha \text{ simple}} k_{\alpha} \alpha. \quad (6.77)$$

This follows by induction: It is obviously true for the simple roots themselves. Any other positive root can be written as a sum of positive roots, hence the statement follows. Since the roots are linearly independent, the decomposition is unique, and we can associate to any positive root its *level* $k = \sum_{\alpha} k_{\alpha}$.

Then we have $\vec{\xi} \cdot \vec{\alpha} = 0$ for all roots α , so we see that the operator $\vec{\xi} \cdot \vec{H}$ commutes with all elements of the algebra,

$$\left[\vec{\xi} \cdot \vec{H}, H_i \right] = \left[\vec{\xi} \cdot \vec{H}, E_{\alpha} \right] = 0. \quad (6.78)$$

But this means that $\vec{\xi} \cdot \vec{H}$ is in the center of the algebra, which is trivial for a semisimple algebra. Hence there is no such ξ , and the simple roots form a basis of \mathbb{R}^r . Hence, in particular, the number of simple roots is equal to the rank of the algebra.

- Fifth, we can find all positive roots. That is, given the simple roots, we can determine whether a linear combination $\gamma_k = \sum k_\alpha \alpha$ is a root or not. The way is by induction over the level and using the master formula again. The key point is that for the simple roots, i.e. those at level one, all $q^i = 0$ since the difference of simple roots is never a root. Hence from the master formula we can find the p^i , and thus the allowed roots on level two. Now for these roots, we by construction know the q^i , hence we again can find the p^i , and continue this process until we found all the roots, i.e. until at some level all roots have all $p^i = 0$.
- We can even go further and reconstruct the full algebra from the simple roots: Given a set of simple roots α^i , we can find all positive roots α . Then we know the algebra consists of the Cartan generators H_i , raising operators E_α and lowering operators $E_{-\alpha}$. All commutators are now fixed as in Eq. (6.50) except for the numbers $e_{\alpha,\beta}$. They are determined by considering the action of each $\mathfrak{su}(2)$ subalgebra

6.5.1 Example: $\mathfrak{su}(3)$

From Eq. (6.65), we can read off the positive roots of $\mathfrak{su}(3)$. One of them, namely $(1, 0)$, is the sum of the other two, which are the simple roots, i.e. at level $k = 1$:

$$\alpha^1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad \alpha^2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2} \right). \quad (6.79)$$

To see what we can add, we use the master formula (and use that α^1 and α^2 have length one). We can also save some work by remembering that 2α is never a root, so we only have to check whether we can add α^2 to α^1 , and we find

$$\vec{\alpha}^1 \cdot \vec{\alpha}^2 = -\frac{1}{2} = -\frac{1}{2}p \quad \Rightarrow \quad p = 1. \quad (6.80)$$

Hence $\alpha^1 + \alpha^2$ is a root, and the only one at level 2. We cannot add any more to it, since

$$(\vec{\alpha}^1 + \vec{\alpha}^2) \cdot \vec{\alpha}^1 = (\vec{\alpha}^1 + \vec{\alpha}^2) \cdot \vec{\alpha}^2 = \frac{1}{2} = -\frac{1}{2}(p - q). \quad (6.81)$$

Clearly, $q = 1$ for both cases, so $p = 0$. Therefore α^1 , α^2 and $\alpha^1 + \alpha^2$ are the only positive roots of $\mathfrak{su}(3)$.

6.5.2 Constructing the Algebra

To show that the root system contains as much information as the algebra, one should reconstruct the algebra from the roots. We will not do this explicitly, but only sketch the idea.

Once we know the simple roots, we can construct all roots. In the adjoint representation, each root correspond to exactly one state. For this state, we can determine its spin $j = p + q$ under the $\mathfrak{su}(2)$ subalgebra associated to any root α , so we can determine the action of the E_{\pm} operators. Now all that is left is to find the commutators $[E_{\alpha_1}, E_{\alpha_2}]$, which in turn is fixed by the Jacobi identity in terms of scalar products of roots.

6.6 Representations and Fundamental Weights

6.6.1 Highest Weight Construction

We will now discuss a convenient way of fixing a representation in terms of a particular set of weight vectors. Assume a simple Lie algebra with rank r and simple roots α^i . Then we want to construct representations by some highest weight construction like we did for $\mathfrak{su}(2)$. Since now we can distinguish raising and lowering operators, we can define a highest weight: A weight μ is called a highest weight if $\mu + \alpha$ is not a weight for any positive root α . This is equivalent to saying that

$$E_{\alpha} |\mu\rangle = 0 \tag{6.82}$$

for all positive roots α . Then we can obtain every state in the representation by acting on $|\mu\rangle$ with lowering operators $E_{-\alpha}$, where α is some positive root. This also shows that in any irreducible representation, there is only one state of weight μ , because otherwise the representation would fall apart into disjoint copies.

Actually, we don't need all roots, but only the simple ones, because

- a) the positive roots are sums of the simple ones, so μ is a highest weight exactly if $\mu + \alpha_i$ is not a weight for all simple roots α^i , and
- b) all lowering operators can be expressed as commutators of those of the simple roots, since e.g.

$$E_{\alpha^i + \alpha^j} \sim [E_{\alpha^i}, E_{\alpha^j}] \tag{6.83}$$

Hence in the master formula, the highest weights have $p = 0$ for all α_i ,

$$\frac{\vec{\mu} \cdot \alpha^i}{\vec{\alpha}^i \cdot \vec{\alpha}^i} = \frac{1}{2} q^i. \tag{6.84}$$

So the highest weight is determined by r integers $q_i \geq 0$, and in turn, every set of such numbers determines a highest weight, i.e. every such weight can be used as a starting point for an irreducible representation. The q^i are called the *Dynkin coefficients*, and a representation with this highest weight is labelled by $D = (q^1, \dots, q^r)$. Hence, we can find all representations of a given algebra just from the simple roots.

6.6.2 Fundamental Weights

Since the Dynkin coefficients specify the representation, it is then natural to consider a set of r special “basis” highest weights μ^i which satisfy

$$\frac{\vec{\mu}^i \cdot \vec{\alpha}^j}{\vec{\alpha}^j \cdot \vec{\alpha}^j} = \frac{1}{2} \delta^{ij}. \quad (6.85)$$

These weights are called *fundamental weights*, and the representations with these highest weights are the *fundamental representations* $D^i = (0, \dots, 0, 1, 0, \dots, 0)$.

Every highest weight can be expressed as an integral linear combination of the fundamental weights,

$$\mu = \sum_i q^i \mu^i. \quad (6.86)$$

Hence, every representation can be obtained from a tensor product of fundamental representations. This tensor product will in general not be irreducible, i.e. its set of weights will contain several highest weights. We can then reconstruct the desired irreducible representation by applying lowering operators. As long as we just care for the states, however, we can directly work with the Dynkin labels: If we start with highest weight with Dynkin labels q^i , we can descend in direction of root α^i q^i times. This gives us further states. Since we know the p^i of all of these states, we can easily find the further q^i 's, and complete the full representation. As an example, we will consider $\mathfrak{su}(3)$ again:

Example: $\mathfrak{su}(3)$

The simple roots of $\mathfrak{su}(3)$ are given in Eq. (6.79). The fundamental weights are solutions to Eq. (6.85), and are easily seen to be

$$\mu^1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}} \right), \quad \mu^2 = \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}} \right). \quad (6.87)$$

It is customary to label each weight by its $q - p$ values, which, for the highest weight, coincides with the Dynkin coefficients. To distinguish this from other r -tuples, it is enclosed in a box, such that e.g. the fundamental weight μ^1 corresponds to $\boxed{1 \ 0}$. Each time we subtract a simple root α^j from a weight, the $p^i - q^i$ values change by

$$-2 \frac{\vec{\alpha}^j \cdot \vec{\alpha}^i}{\vec{\alpha}^i \cdot \vec{\alpha}^i}. \quad (6.88)$$

For $\mathfrak{su}(3)$, this means that lowering by the simple roots corresponds to subtracting the following numbers:

$$\begin{aligned} \alpha^1 &\leftrightarrow \boxed{2 \ -1} \\ \alpha^2 &\leftrightarrow \boxed{-1 \ 2} \end{aligned} \quad (6.89)$$

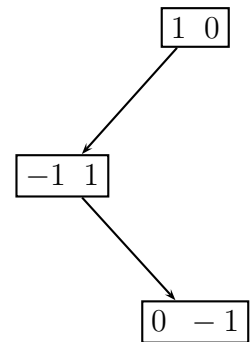


Figure 6.4: Weights of the $(1, 0)$ representation.

Let us consider the two fundamental representations. It is convenient to write down the scalar products involved:

$$\bar{\alpha}^i \cdot \bar{\alpha}^i = 1, \quad \bar{\alpha}^1 \cdot \bar{\alpha}^2 = -\frac{1}{2}, \quad \bar{\alpha}^i \cdot \bar{\mu}^j = \frac{1}{2} \delta^{ij}. \quad (6.90)$$

For the representation with Dynkin labels $(1, 0)$, we start with weight μ^1 and can descend exactly once in direction α^1 to obtain the weight $\boxed{-1 \ 1}$, since for the highest weight, the Dynkin labels are just the q values. In the diagram, this is denoted by an arrow going down and to the left. We cannot go in that direction again. For the α^2 direction, we see that $q - p = 1$, and that $p = 0$ since we arrived there by descending by α^1 . Hence, $q = 1$ and we can go in the α^2 direction exactly once, arriving at $\boxed{0 \ -1}$. From this weight, we cannot go further: For the α^1 direction, we have $p = 0$ and $q - p = 0$, so $q = 0$, while for α^2 we have $p = 1$ and $q - p = -1$, so again $q = 0$. (We also knew this, because we had $q = 1$ for the second weight.)

In summary, the $D^1 = (1, 0)$ representation contains the three weights μ^1 , $\mu^1 - \alpha^1$ and $\mu^1 - \alpha^1 - \alpha^2$, which are the weights of the vector representation (6.67). By a very similar discussion, we can analyse the representation $D^2 = (0, 1)$. Its weight diagram is basically the mirror image of Fig. 6.4, and it gives the complex conjugate representation $\bar{\mathbf{3}}$.

As another example, consider the representation with Dynkin labels $(1, 1)$. In the first step, we can go down in both directions once. When going in the α^1 direction, we arrive at $\boxed{-1 \ 2}$ – these are the $q - p$ values, while the p 's are $(1, 0)$. So the q 's are the sum, $q = q - p + p = (0, 2)$, and we can go down twice in α^2 direction to arrive at $\boxed{1 \ -2}$, whence we can finally descend to $\boxed{-1 \ -1}$. The diagram is symmetric about the vertical axis because the initial state is invariant under interchange of the roots, and the roots correspond to symmetric $q - p$'s.

Note that in between, we pass through $\boxed{0 \ 0}$ – this state is orthogonal to both simple roots. Since they are a basis, this state has weight zero, and so the two states above it have the two simple roots as weights. Hence, this representation is actually the adjoint, and $\boxed{0 \ 0}$ corresponds to the Cartan generators. This shows that a particular weight might have several associated states. There is a general way to determine the degeneracy of a weight from Freudenthal's recursion formula, which also leads to an explicit expression for the dimension of a representation given its highest weight, the Weyl dimension formula. They are discussed e.g. in the books by Cahn or by Fuchs and Schweigert.

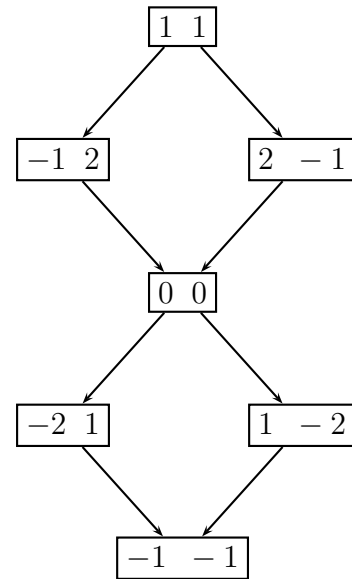


Figure 6.5: Weights of the $(1, 1)$ representation.

6.7 Cartan Matrix, Dynkin Diagrams

Now that we have seen that the simple roots contain all the information of the Lie algebra, we introduce two ways of displaying the properties of simple roots: The Cartan matrix and the Dynkin diagram. They are equivalent to each other and to the set of simple roots in the sense that from any one one can construct the other two. However, the Cartan matrix is more convenient for building representations, while the Dynkin diagram clearly shows the structure of the algebra and possible subalgebras. In the following, we again denote the simple roots by α^i , where i runs from 1 to the rank r .

6.7.1 The Cartan Matrix

When building representations, we saw that the expression 6.88 is rather convenient. It is important enough to get its own name: For a given Lie algebra of rank r , the *Cartan matrix* A is an $r \times r$ matrix with entries

$$A^{ij} = 2 \frac{\vec{\alpha}^i \cdot \vec{\alpha}^j}{\vec{\alpha}^j \cdot \vec{\alpha}^j}. \quad (6.91)$$

We immediately deduce a few properties:

1. A is invertible, since the simple roots are linearly independent.
2. The diagonal entries are $A^{ii} = 2$.
3. In general, A is not symmetric due to the denominator. However, when $A^{ij} = 0$, then also $A^{ji} = 0$.
4. The off-diagonal entries can only be $0, \pm 1, \pm 2, \pm 3$ because of Table 6.1. Furthermore, from Eq. (6.71) we see that the values cannot be positive, so they must be $0, -1, -2$ or -3 .
5. We deduce for the product $A^{ij}A^{ji}$ (no sum over i, j !)

$$A^{ij}A^{ji} = 4 \frac{(\vec{\alpha}^i \cdot \vec{\alpha}^j)^2}{\vec{\alpha}^{i2}\vec{\alpha}^{j2}} = 4 \cos^2 \phi \leq 4. \quad (6.92)$$

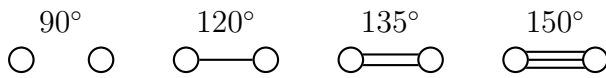
The inequality is strict for $i \neq j$ because then α^i and α^j are linearly independent. Hence, if $A^{ij} = -2$ or $A^{ij} = -3$, then $A^{ji} = -1$

Each positive root defines an $\mathfrak{su}(2)$ inside the algebra. You can think of the Cartan matrix as telling you how the $\mathfrak{su}(2)$ s mix, i.e. how the simple roots transform under the other roots' $\mathfrak{su}(2)$ s. This is the reason why the Cartan matrix helps to construct general representation starting from a highest weight.

6.7.2 Dynkin Diagrams

Now we come to the other description of the algebra, this time in form of pictures. Recall that for simple roots, the allowed angles in Table 6.1 are more restricted to be 90° , 120° , 135° or 150° , while the relative lengths are still as in the table. We define the *Dynkin diagram* of a Lie algebra by the following procedure:

0. A Dynkin diagram consists of open circles, filled circles and lines between circles.
1. For each simple root α^i , draw an open circle.
2. Connect two circles with zero, one, two or three lines if the angle between the roots is 90° , 120° , 135° or 150° , respectively:



Alternatively, the number of lines can be determined from the Cartan matrix: For roots α^i and α^j , the number of lines is $A^{ij}A^{ji}$.

3. Roots with an angle of 135° or 150° are of different length. Fill the circle corresponding to the shorter root.

Alternatively, if $A^{ij}A^{ji} = 2$ or $A^{ij}A^{ji} = 3$, fill out the circle for root α^i if $A^{ij} > A^{ji}$ and α^j otherwise.

For example, we have seen that $\mathfrak{su}(2)$ has rank one, and thus its Dynkin diagram is very simple: $\mathfrak{su}(2) \cong \bigcirc$. For $\mathfrak{su}(3)$, we have two simple roots with an angle of 120° , so $\mathfrak{su}(3) \cong \bigcirc - \bigcirc$.

6.8 Classification of Simple Lie Algebras

The Dynkin diagram allows us to classify all possible simple Lie algebras by translating the algebraic properties into geometrical and diagrammatical ones.

6.8.1 Π Systems

Let us consider the conditions a set of vectors $\{\alpha^i\}$ needs to satisfy in order to qualify as a set of roots. It turns out the following conditions are sufficient:

1. The α^i are linearly independent.
2. For two distinct vectors α and β , $2\alpha \cdot \beta/\alpha^2$ is a nonpositive integer.
3. The set of vectors is indecomposable, i.e. it cannot be separated into two orthogonal subsets.

If these conditions are satisfied, the set of vectors is called a Π -system. Recall that from condition 2. you can deduce that the nonpositive integer is actually 0, -1 , -2 or -3 , and that the connection between the angle and the relative length still holds. The third condition ensures that the algebra is simple – if the set of roots splits into orthogonal pairs, you can show that the algebra splits into invariant subalgebras, so it is semisimple and generates a direct product group.

Given a Π system, we can draw the Dynkin diagram – they are equivalent (up to a normalisation). What do the conditions mean in terms of the diagram? In other words, what are acceptable Dynkin diagrams? The third condition implies that the diagram cannot be separated into two diagrams without cutting a line. The second condition is automatically satisfied since the number of lines between two circles is just minus the nonnegative integer. The only condition that requires some work is the first, and it will be rather restrictive.

6.8.2 Constraints

One immediate consequence of the conditions for a Π system is that any connected subset of a Π system is again a valid Π system. This will turn out to be very important, because everything we find for Dynkin diagrams with few nodes carries over to all subdiagrams of larger ones. Hence we will be able to exclude a huge number of prospective diagrams because they contain non-admissible subdiagrams.

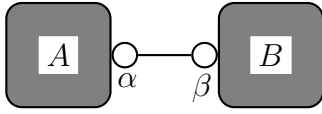
Let us first find the allowed diagrams for one to three nodes:

- i. For one vector, there obviously is one diagram: \bigcirc
- ii. For two vectors, there are three possible diagrams: $\bigcirc\text{---}\bigcirc$, $\bigcirc\text{=}\bigcirc$, and $\text{=}\bigcirc$.
- iii. For three vectors, linear independence kicks in the first time: Three vectors are independent unless they lie in a plane, in which case the sum of angles between them is equal to 360° . So the allowed diagrams cannot have three single lines, or two double lines, or one single and one triple line. Furthermore, if the sum of the angles is larger than 360° , we can switch to some opposite angle to get a sum smaller than 360° . Hence the only allowed ones are $\bigcirc\text{---}\bigcirc\text{---}\bigcirc$ and $\bigcirc\text{---}\bigcirc\text{=}\bigcirc$

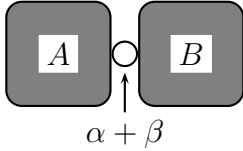
In particular, the argument implies that in a three-vector diagram there can be no triple line. Since a connected subdiagram is again a valid diagram, we see that no diagram with three or more nodes can contain a triple line, so $\text{=}\bigcirc$ is the only one containing a triple line.

Next, there are two ways how we can form a new diagram from a given one, shrinking lines and collapsing branches:

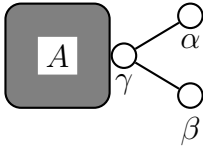
1. If this



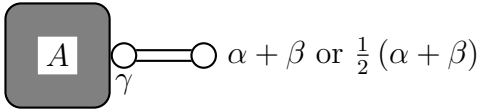
is a valid diagram (where A and B are arbitrary subdiagrams), then so is this:



2. If this



is a valid diagram, then so is this:



For the proof, we have to check that the reduced system again satisfies the properties of a Π system. Clearly, conditions 1 and 3 are still satisfied, so we just have to check condition 2:

1. In the first case, α , β and $\alpha + \beta$ have the same length (since $\cos 120^\circ = -\frac{1}{2}$). Furthermore, there is no vector which is connected to both α and β , so any vector $\gamma \in A$ is orthogonal to β , and any $\gamma' \in B$ is orthogonal to α . Hence

$$\gamma \cdot (\alpha + \beta) = \gamma \cdot \alpha \quad \text{and} \quad \gamma' \cdot (\alpha + \beta) = \gamma' \cdot \beta. \quad (6.93)$$

So the set $A \cup B \cup \{\alpha + \beta\}$ satisfies all the axioms for a Π system if $A \cup B \cup \{\alpha, \beta\}$ does.

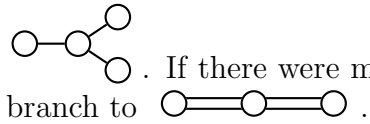
2. We see that α , β and γ have the same length, and that $\alpha \cdot \beta = 0$, while $\alpha \cdot \gamma = \gamma \cdot \beta = -\frac{1}{2}\alpha^2$. Then $(\alpha + \beta)^2 = 2\alpha^2$, and we have that

$$\begin{aligned} 2 \frac{\gamma \cdot (\alpha + \beta)}{\gamma^2} &= -2, \\ 2 \frac{\gamma \cdot (\alpha + \beta)}{(\alpha + \beta)^2} &= -1. \end{aligned} \quad (6.94)$$

Hence $A \cup \{\gamma, \alpha + \beta\}$ is a Π system if $A \cup \{\gamma, \alpha, \beta\}$ is. Here γ is the shorter root, so in the associated diagram, its node should be black. A similar conclusion applies to the set $A \cup \{\gamma, \frac{1}{2}(\alpha + \beta)\}$ – here the new root is the blackened one.

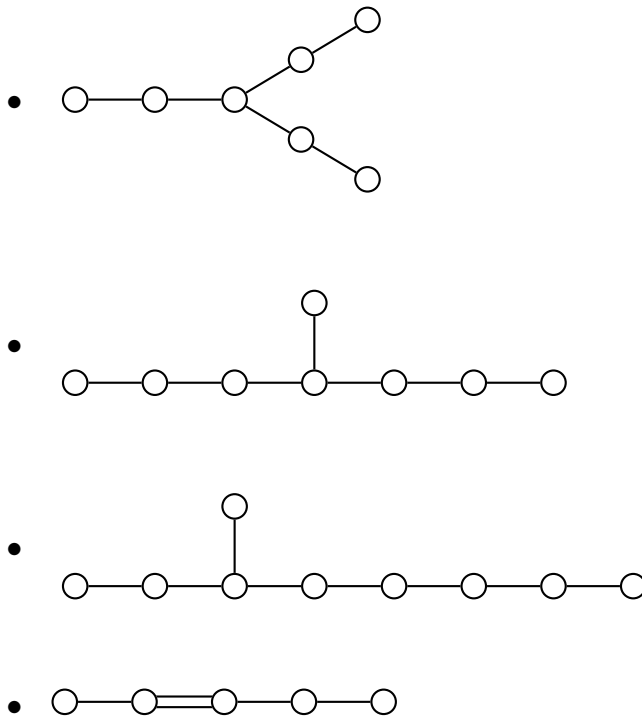
From this follow a few corollaries:

1. No diagram contains more than one double line: Since there can be no triple lines (for rank $r > 2$), the diagram contains only single and double lines. If there was more than one double line, we could shrink away the single lines and arrive at the three-vector subsystem $\bigcirc = \bigcirc = \bigcirc$, which is not a valid diagram.
2. No diagram contains a closed loop: That could be shrunk to a closed loop of three vectors, which again is not allowed.
3. Any branch point is of the form of three single lines out of one node,



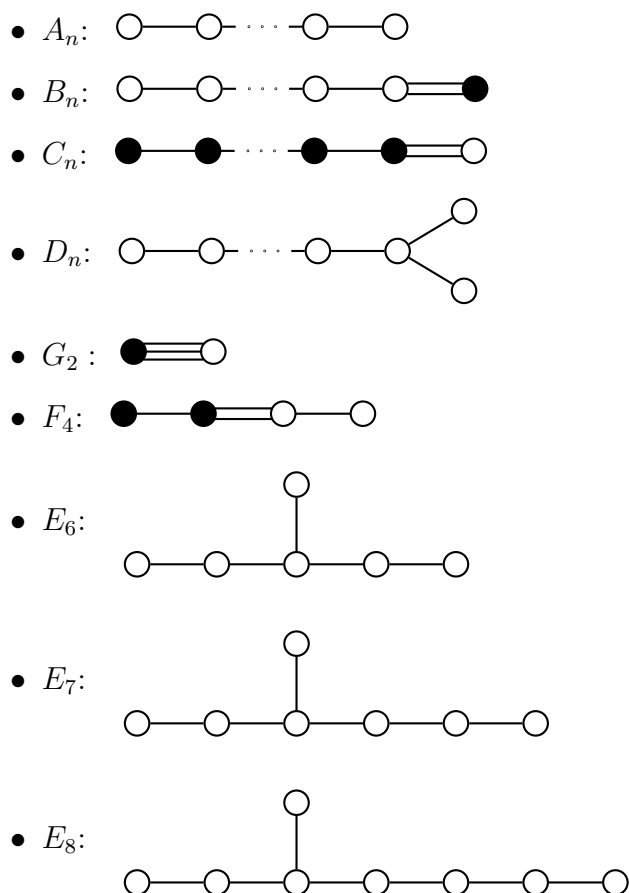
4. No diagram contains two branch points.
5. No diagram contains both a branch point and a double line.

Finally, we can exclude a number of individual diagrams: No Π -system can contain any of the following diagrams:



This follows because the vectors would not be linearly independent – you can either find linear combinations that vanish (this is possible because you know all scalar products of the vectors, up to some overall length), or you can check that the Cartan matrix has zero determinant. (This holds independently of the relative lengths in the last diagram). Hence we also see that if there is a double line or a branch point, they need to be basically at the end of the diagram, except for diagrams with only very few nodes.

Now we have arrived at the classification theorem of simple Lie algebras: Any simple Lie algebra must either belong to one of four infinite families, which are called A_n to D_n , or one of five *exceptional Lie algebras*, G_2 , F_4 , E_6 , E_7 or E_8 . The Dynkin diagrams and the names of the algebras are (where A_n contains n nodes etc., and the black nodes correspond to shorter roots):



Note that for low enough rank, there are some redundancies. There is only one rank-one algebra, $\mathfrak{su}(2)$, so $A_1 = B_1 = C_1$. In rank two, we see that $B_2 = C_2$. For the D_n series, you remove nodes from the left, so $D_2 = A_1 \times A_1$ and $D_3 = A_3$. However, these equivalences are accidental and do not continue for higher rank.

As usual, we have discussed complex Lie algebras. All the algebras have various real forms. In particular, the infinite families A_n to D_n have as their compact real form the Lie algebras of the *classical groups* $SU(n + 1)$, $SO(2n + 1)$, $Sp(2n)$ and $SO(2n)$. We will discuss these in the next section.

6.9 The Dynkin Diagrams of the Classical Groups

The Lie groups $SO(n)$, $SU(n)$ and $Sp(n)$ are called classical groups. For $SO(n)$ and $SU(n)$ we have discussed the Lie algebras in Section 5.4.2. Here we will go into more detail and find the Dynkin diagrams. We will use the physicists' convention and use Hermitean matrices as the elements of the algebras.

To find the simple roots, it will suffice to find a convenient explicit form of the Cartan elements, because then we know the weights of the defining representation, and the differences of the weights will be the roots. The Cartan elements must be Hermitean, and (almost) diagonal, so they can be found rather straightforwardly. We will only present the results.

6.9.1 $\mathfrak{su}(n)$

The Lie algebra $\mathfrak{su}(n)$ consists of $n \times n$ traceless Hermitean matrices. Its (real) dimension is $n^2 - 1$. To find the rank, use that we can simultaneously diagonalise all elements of the Cartan subalgebra, and their diagonal elements are real. Hence the rank is the dimension of real traceless $n \times n$ matrices, which is $n - 1$.

Cartan Generators

The usual convention is to normalise the generators T_a such that $\text{tr} T_a T_b = \frac{1}{2} \delta_{ab}$. The Cartan elements H_m for $m = 1, \dots, n - 1$ are conveniently chosen diagonal,

$$H_m = \frac{1}{\sqrt{2m(m+1)}} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -m & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}. \quad (6.95)$$

The explicit components are

$$(H_m)_{ij} = \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^m \delta_{ik} \delta_{jk} - m \delta_{im+1} \delta_{jm+1} \right). \quad (6.96)$$

As basis states of the defining representation we can choose the standard basis e^i of \mathbb{C}^n . The weights are in general

$$\omega^i = ((H_1)_{ii}, (H_2)_{ii}, \dots, (H_{n-1})_{ii}). \quad (6.97)$$

Explicitly, we have

$$\omega^1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \dots, \frac{1}{\sqrt{2m(m+1)}}, \dots, \frac{1}{\sqrt{2n(n-1)}} \right), \quad (6.98a)$$

$$\omega^2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \dots, \frac{1}{\sqrt{2m(m+1)}}, \dots, \frac{1}{\sqrt{2n(n-1)}} \right), \quad (6.98b)$$

$$\omega^3 = \left(0, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{6}}, \dots, \frac{1}{\sqrt{2m(m+1)}}, \dots, \frac{1}{\sqrt{2n(n-1)}} \right), \quad (6.98c)$$

\vdots

$$\omega^{m+1} = \left(\underbrace{0, \dots, 0}_{m-1 \text{ times}}, -\frac{m}{\sqrt{2m(m+1)}}, \dots, \frac{1}{\sqrt{2n(n-1)}} \right), \quad (6.98d)$$

\vdots

$$\omega^n = \left(0, \dots, \dots, 0, \frac{-n+1}{\sqrt{2n(n-1)}} \right) \quad (6.98e)$$

You can check that the weights have the same length and the same scalar product,

$$(\vec{\omega}^i)^2 = \frac{n-1}{2n}, \quad \vec{\omega}^i \cdot \vec{\omega}^j = -\frac{1}{2n}. \quad (6.99)$$

Furthermore, the tracelessness of the generators leads to

$$\sum_i \omega^i = 0. \quad (6.100)$$

This was to be expected – we have n weights, but they have only $n-1$ components.

This choice of Cartan generators and weights has the nice feature that we very easily get the positive and simple roots, but we unfortunately have to define a nonstandard ordering: A weight is positive if the *last* nonvanishing component is positive. Using this ordering prescription, the weights are already ordered,

$$\omega^1 > \omega^2 > \dots > \omega^n. \quad (6.101)$$

Roots

The roots are the differences of weights, $\omega^i - \omega^j$. Due to the ordering, the positive roots are $\omega^i - \omega^j$ for $i < j$. If there is a k such that $i < k < j$, we have

$$\omega^i - \omega^j = \underbrace{\omega^i - \omega^k}_{\text{positive}} + \underbrace{\omega^k - \omega^j}_{\text{positive}}, \quad (6.102)$$

and thus $\omega^i - \omega^j$ is not simple. So the simple roots finally are

$$\alpha^i = \omega^i - \omega^{i+1}. \quad (6.103)$$

From Eq. (6.99), we can deduce

$$\vec{\alpha}^i \cdot \vec{\alpha}^i = 1, \quad \vec{\alpha}^i \cdot \vec{\alpha}^{i+1} = -\frac{1}{2}, \quad \vec{\alpha}^i \cdot \vec{\alpha}^j = 0 \quad \text{for } i \neq j, j \pm 1. \quad (6.104)$$

In other words, all roots have the same length, neighbouring roots enclose an angle of 120° , and other roots are orthogonal. Hence the Dynkin diagram for $\mathfrak{su}(n)$ is A_{n-1} ,

$$\mathfrak{su}(n) \cong \begin{array}{cccc} \alpha^1 & \alpha^2 & \dots & \alpha^{n-2} & \alpha^{n-1} \\ \circ & \circ & \dots & \circ & \circ \end{array} . \quad (6.105)$$

Equivalently, the Cartan matrix is

$$A_{\mathfrak{su}(n)} = \begin{pmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} . \quad (6.106)$$

Fundamental Weights

The fundamental weights μ^i have to satisfy (since $\vec{\alpha}^i \cdot \vec{\alpha}^i = 1$)

$$\vec{\alpha}^i \cdot \vec{\mu}^j = \frac{1}{2} \delta^{ij} \quad (6.107)$$

A straightforward calculation shows they are

$$\begin{aligned} \mu^1 &= \omega^1, \\ \mu^2 &= \omega^1 + \omega^2, \\ &\vdots \\ \mu^{n-1} &= \omega^1 + \omega^2 + \dots + \omega^{n-1} \end{aligned} \quad (6.108)$$

Due to Eq. (6.100), the last fundamental weight is $\mu^{n-1} = -\omega^n$. Note that lowest weight of the μ^1 representation is

$$\mu^1 - \alpha^1 - \dots - \alpha^{n-1} = \omega^n, \quad (6.109)$$

i.e. minus the highest weight of $D(\mu^{n-1})$. Hence, the representations are conjugate to each other, $\overline{D(\mu^1)} = D(\mu^{n-1})$.

So now we can in principle determine any representation of $\mathfrak{su}(n)$ from its Dynkin coefficients as a product of the fundamental representations. Actually, for $\mathfrak{su}(n)$ there is one more simplification: We only really need the defining representation, which is usually denoted by its dimension as \mathbf{n} . To add some confusion, the defining representation is often called *the* fundamental representation, and its conjugate is called the

antifundamental. We will not follow this usage. This comes about because all other representation can be obtained from tensor products of this representation with itself. To see this, recall that a tensor product of representations is in general reducible. One irreducible piece for the tensor product of a representation with itself is the antisymmetric combination: Label the states in the defining representation by $|\mathbf{n}, i\rangle$ with $i = 1, \dots, n$. Then the general tensor product is spanned by states $|\mathbf{n}, i\rangle \otimes |\mathbf{n}, j\rangle$. The antisymmetric piece is

$$(\mathbf{n} \otimes \mathbf{n})_a = \{A^{ij} |\mathbf{n}, i\rangle \otimes |\mathbf{n}, j\rangle\} \quad (6.110)$$

where $A^{ij} = -A^{ji}$. Since weights of tensor products add, the highest weight is the sum of the highest and second-highest weights of the $|\mathbf{n}\rangle$, which are μ^1 and $\mu^1 - \alpha^1$. Hence the highest weight of the antisymmetric tensor product is

$$\mu^1 + (\mu^1 - \alpha^1) = \omega^1 + \omega^2 = \mu^2. \quad (6.111)$$

Hence (since the highest weight is unique), this is precisely the second fundamental representation! This pattern continues, so that we have

$$D(\mu^k) = \underbrace{(\mathbf{n} \otimes \dots \otimes \mathbf{n})}_k \text{ times} \quad (6.112)$$

In particular, its dimension is

$$\dim D(\mu^k) = \binom{n}{k} \quad (6.113)$$

In Section 6.11 we will discuss how to use Young tableaux to find representations of $\mathfrak{su}(n)$.

6.9.2 $\mathfrak{so}(n)$

For $\mathfrak{so}(n)$, we can go through a similar analysis: The algebra elements are the imaginary antisymmetric $n \times n$ matrices. Hence we cannot choose diagonal Cartan generators, but we can come close: We choose them block-diagonal with 2×2 blocks, of which all but one are zero (if n is odd, there is one more row and column of zeroes):

$$H_m = \begin{pmatrix} 0_2 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 0_2 & & & & & & & \\ & & & \sigma^2 & & & & & & \\ & & & & 0_2 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 0_2 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & 0_2 & \end{pmatrix} \quad \text{position } m \quad (6.114)$$

Explicitly, the components are

$$(H_m)_{ij} = -i(\delta_{i,2m-1}\delta_{j,2m} - \delta_{i,2m}\delta_{j,2m-1}). \quad (6.115)$$

They still commute, and there are no more commuting algebra elements. So the rank of $\mathfrak{so}(n)$ is $n/2$ if n is even and $(n-1)/2$ if n is odd. We will denote the rank by k . The eigenvectors of the H_m are not the standard basis of \mathbb{C}^n , but are of the form $(0, \dots, 0, 1, \pm i, 0, \dots, 0)$, and the associated eigenvalues are ± 1 . Hence the weights of the defining representation are k -component vectors commonly denoted e^i (because they look like the standard basis of \mathbb{C}^k):

$$\pm e^i = (0, \dots, 0, \pm 1, 0, \dots, 0) \quad (6.116)$$

\uparrow position i

Clearly they satisfy $\vec{e}^i \cdot \vec{e}^j = \delta^{ij}$. Also, we see that for every weight, its negative is also a weight, so the defining representation of $\mathfrak{so}(n)$ is self-conjugate. Here we can return to our old definition of positivity and call the e^i positive and $-e^i$ negative weights. Note that for odd n , the last row and column of the Cartan elements are zero, so that the n -component vector $(0, \dots, 0, 1)$ has weight zero, which was not a weight for even n . This means that to find the roots, one has to distinguish even and odd n .

Roots of $\mathfrak{so}(2k)$

Let us first consider even $n = 2k$. Then all weights are of the form $\pm e^i$, and their differences, i.e. the roots, are of the form

$$\pm e^i \mp e^j, \quad i \neq j. \quad (6.117)$$

The constraint $i \neq j$ comes about because if $i = j$, the root would either be zero, or the corresponding generator would map the state with weight e^i to the one with $-e^i$. But this would mean that the operator would itself be block-diagonal and antisymmetric, and you can convince yourself that this operator does not exist. Hence there are $2 \times 2 \times \binom{k}{2}$ roots. Note that this is consistent with the dimension of $\mathfrak{so}(2k)$:

$$\dim(\mathfrak{so}(2k)) = \frac{2k(2k-1)}{2} = 2 \times 2 \times \binom{k}{2} + k. \quad (6.118)$$

Our (traditional first-component based) definition of positivity implies that the positive roots are $e^i \pm e^j$ for $i < j$ (this restriction avoids double-counting for the plus sign). The simple roots are then, by a similar argument as above, the differences of neighbouring weights, and the sum of the two last ones:

$$\alpha^i = e^i - e^{i+1}, \quad i = 1, \dots, k-1, \quad \alpha^k = e^{k-1} + e^k. \quad (6.119)$$

To draw the Dynkin diagram, we compute the scalar products. The nonvanishing ones are (again with $i = 1, \dots, k-1$)

$$\vec{\alpha}^i \cdot \vec{\alpha}^i = \vec{\alpha}^k \cdot \vec{\alpha}^k = 2, \quad \vec{\alpha}^{i-1} \cdot \vec{\alpha}^i = -1, \quad \vec{\alpha}^{k-2} \cdot \vec{\alpha}^k = -1. \quad (6.120)$$

Hence the diagram is D_k ,

$$\mathfrak{so}(2k) \cong \begin{array}{ccccccc} & \alpha^1 & \alpha^2 & \cdots & \alpha^{k-3} & \alpha^{k-2} & \alpha^k \\ & \circ & \circ & \cdots & \circ & \circ & \circ \\ & | & | & & | & / & \backslash \\ & & & & & \circ & \circ \\ & & & & & \alpha^{k-1} & \end{array} . \quad (6.121)$$

The Cartan matrix is

$$A_{\mathfrak{so}(2k)} = \begin{pmatrix} 2 & -1 & 0 & & & & \\ -1 & 2 & -1 & & & & \\ 0 & -1 & 2 & & & & \\ & & & \ddots & & & \\ & & & & 2 & -1 & 0 & 0 \\ & & & & -1 & 2 & -1 & -1 \\ & & & & 0 & -1 & 2 & 0 \\ & & & & 0 & -1 & 0 & 2 \end{pmatrix} . \quad (6.122)$$

Roots of $\mathfrak{so}(2k + 1)$

For odd $n = 2k + 1$, we have to modify the analysis a bit because now also zero is a weight. Hence we get a new set of roots, namely all $\pm e^i$. This again matches the dimension, since $\dim(\mathfrak{so}(2k + 1)) - \dim(\mathfrak{so}(2k)) = 2k$. Hence, α^k is not anymore a simple root, and its place is taken by $\alpha^k = e^k$. This root is shorter than the other ones, and its only nonvanishing scalar product is

$$\vec{\alpha}^{k-1} \cdot \vec{\alpha}^k = -1 . \quad (6.123)$$

Hence the Dynkin diagram becomes that of B_k ,

$$\mathfrak{so}(2k + 1) = \begin{array}{ccccccc} & \alpha^1 & \alpha^2 & \cdots & \alpha^{k-2} & \alpha^{k-1} & \alpha^k \\ & \circ & \circ & \cdots & \circ & \circ & \bullet \\ & | & | & & | & || & \\ & & & & & & \end{array} . \quad (6.124)$$

Here the Cartan matrix is not symmetric,

$$A_{\mathfrak{so}(2k+1)} = \begin{pmatrix} 2 & -1 & 0 & & & & \\ -1 & 2 & -1 & & & & \\ 0 & -1 & 2 & & & & \\ & & & \ddots & & & \\ & & & & 2 & -2 & \\ & & & & -1 & 2 & \end{pmatrix} . \quad (6.125)$$

Fundamental Weights of $\mathfrak{so}(2k)$

The fundamental weights can be constructed in the same way as for $\mathfrak{su}(n)$, except for the last two (since the last root is different). We find the fundamental weights

$$\begin{aligned}
 \mu^1 &= e^1, \\
 \mu^2 &= e^1 + e^2, \\
 &\vdots \\
 \mu^{k-2} &= e^1 + \dots + e^{k-2}, \\
 \mu^{k-1} &= \frac{1}{2} (e^1 + \dots + e^{k-1} - e^k), \\
 \mu^k &= \frac{1}{2} (e^1 + \dots + e^{k-1} + e^k).
 \end{aligned} \tag{6.126}$$

As for $\mathfrak{su}(n)$, most of the fundamental representations can be obtained as antisymmetric tensor products of the defining one, which is again denoted by its dimension as \mathbf{n} ,

$$D(\mu^i) = (\mathbf{n} \otimes \dots \otimes \mathbf{n})_a, \quad i = 1, \dots, k-2. \tag{6.127}$$

The last two weights are different – the associated representations are called spinor representations. They cannot be obtained from the defining representation, as is signified by the factor $\frac{1}{2}$, which means they cannot arise as sums of the other weights. Their dimension can be obtained easily, because the weights of the spinors are all of the form

$$\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \dots, \pm \frac{1}{2} \right). \tag{6.128}$$

You can convince yourself that all these weights indeed appear. The two representations are distinguished by the fact that they have either an even or an odd number of minus signs, hence their dimensions are 2^{k-1} .

As the weights are the eigenvalues of the Cartan generators, we also see that group elements

$$g = \exp(i\phi_i H_i), \tag{6.129}$$

have a periodicity of $\phi \sim \phi + 4\pi$ when acting on spinor representations, while the periodicity is $\phi \sim \phi + 2\pi$ when acting on the non-spinorial representations. Hence we see that the spinor representations are the analogues of the half-integral representations of $\mathfrak{su}(2)$ – they are not proper representations of the group $SO(2k)$, but only of its double cover group which is usually denoted $Spin(2k)$. (This is also the universal covering group.) For low enough dimensions, the $Spin(n)$ groups accidentally are given by some classical groups, as e.g. $Spin(3) = SU(2)$ and $Spin(4) = SU(2) \times SU(2)$, but this does not persist in higher dimension.

Fundamental Weights of $\mathfrak{so}(2k+1)$

For odd dimensions, the analysis is slightly modified again. We find the fundamental weights

$$\begin{aligned}\mu^1 &= e^1, \\ \mu^2 &= e^1 + e^2, \\ &\vdots \\ \mu^{k-1} &= e^1 + \cdots + e^{k-1}, \\ \mu^k &= \frac{1}{2}(e^1 + \cdots + e^k).\end{aligned}\tag{6.130}$$

Hence, we see that $\mathfrak{so}(2k+1)$ has one fundamental spinorial representation, which again is only a proper representation of the double cover group $Spin(2k+1)$. The dimension of the spinor is 2^k .

There is a systematic way to work with spinor representations using what is called a Clifford algebra. This is what you use in particle physics under the name of γ matrices, and we will very briefly discuss that in Section 6.13.

6.9.3 $\mathfrak{sp}(2n)$

The last family of classical groups are the *symplectic groups* $Sp(2n)$. They are important in classical mechanics, but not so much in particle physics, so we will be rather brief. They are defined as those $2n \times 2n$ unitary matrices M which preserve an antisymmetric scalar product,

$$M^T J M = J \quad \text{where } J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix},\tag{6.131}$$

Note that there is some confusion in the notation: Sometimes this is called $USp(2n)$, and $Sp(2n)$ is defined without the requirement of unitarity. Also, some people write $Sp(2n)$ as $Sp(n)$.

Clearly the matrices are unitary and have M have $\det M = \pm 1$. Hence the algebra of $Sp(2n)$ consists of Hermitean traceless matrices m which additionally satisfy

$$m^T J + J m = 0.\tag{6.132}$$

A convenient parameterisation is in terms of the Pauli matrices

$$m = \mathbb{1} \otimes iA + \sigma^i \otimes S_i\tag{6.133}$$

with A being real and antisymmetric and S_i real and symmetric. In particular, $\text{tr } A = 0$. Note that there is an $\mathfrak{su}(n)$ subalgebra spanned by

$$\mathbb{1} \otimes iA + \sigma^3 \otimes S_3\tag{6.134}$$

with traceless S_3 . This corresponds to the $SU(n)$ subgroup

$$M = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}. \quad (6.135)$$

It is convenient to choose the Cartan generators as the diagonal matrices. Then the $n - 1$ Cartan generators of the $\mathfrak{su}(n)$ subalgebra are automatically embedded into in the Cartan algebra of $\mathfrak{sp}(n)$ as

$$H_m = \frac{1}{\sqrt{2}} \begin{pmatrix} H_m^{\mathfrak{su}(n)} & 0 \\ 0 & -H_m^{\mathfrak{su}(n)} \end{pmatrix}, \quad m = 1, \dots, n - 1. \quad (6.136)$$

We have included the prefactor to keep the normalisation $\text{tr } H_m H_n = \frac{1}{2} \delta_{mn}$. There is only one remaining diagonal matrix, namely the one where we drop the tracelessness condition:

$$H_n = \frac{1}{2\sqrt{n}} \sigma^3 \otimes \mathbb{1}. \quad (6.137)$$

Thus, the rank of $\mathfrak{sp}(2n)$ is n .

Hence the defining representation has weights which are combinations of $\mathfrak{su}(n)$ weights ω^i and the weight with regard to H_n ,

$$\mu^i = \left(\frac{1}{\sqrt{2}} \omega^i, \frac{1}{2\sqrt{n}} \right) \quad (6.138)$$

and their negatives. So we can find the roots as differences of weights in a manner similar to the $\mathfrak{su}(n)$ case,

$$\alpha^i = e^i - e^{i+1} = \frac{1}{\sqrt{2}} \alpha_{\mathfrak{su}(n)}^i, \quad (6.139)$$

for $i = 1, \dots, n - 1$. Note that now the length of the roots is $\alpha^i \cdot \alpha^i = \frac{1}{2}$. The last simple root is

$$\alpha^n = e^i - (-e^i), \quad (6.140)$$

which has length one. The angles between the first $n - 1$ roots are as for $\mathfrak{su}(n)$, while the last root has scalar products

$$\alpha^n \cdot \alpha^i = 0, \quad i = 1, \dots, n - 2, \quad \alpha^n \cdot \alpha^{n-1} = -\frac{1}{2}. \quad (6.141)$$

Hence the last two roots enclose an angle of 135° , and the Dynkin diagram is C_n ,

$$\mathfrak{sp}(2n) \cong \alpha^1 \quad \alpha^2 \quad \dots \quad \alpha^{n-2} \quad \alpha^{n-1} \quad \alpha^n \quad (6.142)$$

The diagram is basically the “mirror” of the one for $\mathfrak{so}(2k+1)$, and the Cartan matrix is simply the transpose:

$$A_{\mathfrak{sp}(2n)} = \begin{pmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix}. \quad (6.143)$$

6.10 A Note on Complex Representations

At this point we can briefly discuss the issue of complex representations. All of the following discussion applies to the compact real form of the Lie algebras. By complex representations we mean representations that are not equivalent to their complex conjugates. If they are equivalent, i.e. if there is a matrix R such that

$$RT_aR^{-1} = -T_a^*, \quad (6.144)$$

we call the representation real.

Note that we are a bit sloppy here: In mathematical terms, this requirement lumps together “real” and “quaternionic” or “pseudoreal” representations. To understand the difference, it is instructive to consider representations of the group instead of the algebra. The group elements are of the form

$$g = e^{i\phi^a T_a}, \quad (6.145)$$

and clearly the matrix R generates a similarity transformation that takes g to its complex conjugate,

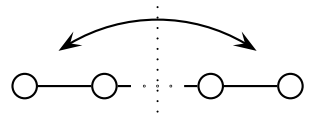
$$RgR^{-1} = e^{i\phi^a RT_aR^{-1}} = e^{-i\phi^a T_a^*} = (e^{i\phi^a T_a})^*. \quad (6.146)$$

Hence, if there is such an R , the representation is equivalent to its complex conjugate. However, it might still be made of complex matrices – an obvious example is the defining representations of $SU(2)$, for which σ^2 is such an R , but which we cannot take real, because then we would end up with the group $U(1)$. Representations with this property are called pseudoreal. A properly real representation is one which can be brought to a purely real form – then we can restrict its action to a real vector space, and this forms a proper real representation. Transferred to the algebra this means that a properly real representation is one which can be brought into purely imaginary form, and a pseudoreal representation is one for which this is not true, but for which an R as above still exists.

Complex conjugation is an algebra automorphism of order two (conjugating twice brings you back to the original representation), and it can be an outer or inner automorphism. If it is an inner automorphism, the representation and its conjugate are

equivalent, and in particular the Dynkin diagrams are unchanged. An outer automorphism, on the other hand, brings you to a non-equivalent representation, and it must be visible as an order-two symmetry of the Dynkin diagram.

We can analyse the Dynkin diagrams of the classical and exceptional algebras to look for possible such symmetries, which indicate that the corresponding algebra might have complex representations. For example, the A_n family of diagrams has a reflection symmetry,



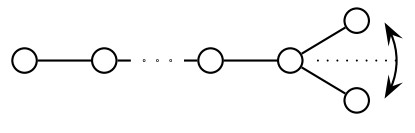
$$(6.147)$$

which maps $\alpha^1 \leftrightarrow \alpha^n$, $\alpha^2 \leftrightarrow \alpha^{n-1}$ and so on. This inverts the Dynkin labels of any representation, so we have

$$\overline{(q_1, \dots, q_n)} = (q_n, \dots, q_1) . \quad (6.148)$$

Hence for $\mathfrak{su}(n)$, a representation is real if its Dynkin labels are invariant under this identification. (Indeed one can show that this automorphism is the complex conjugation.)

For $\mathfrak{so}(2k)$ or D_k , there also is such a symmetry, which only interchanges the last two roots:

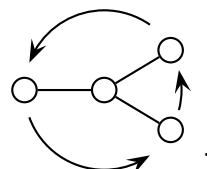


$$(6.149)$$

Hence, also the last two Dynkin labels are interchanged. This matches with what we found before: The defining representation is real, and so are its tensor products, which all have Dynkin labels $(q_1, \dots, q_{k-3}, 0, 0)$. The spinor representations, on the other hand, are mapped into each other. However, this automorphism is not necessarily the complex conjugation – it is for $\mathfrak{so}(4m+2)$, where the spinors are conjugate to each other, but it is not for $\mathfrak{so}(4m)$, where both spinors are self-conjugate.

On the other hand, the B_n and C_n diagrams do not have such a symmetry. Among the exceptional groups, only E_6 has a suitable reflection symmetry, so that it can (and does) have complex representations.

Note finally that there is one example of a higher-order automorphisms, called triality: D_4 (i.e. $\mathfrak{so}(8)$) has a \mathbb{Z}_3 symmetry rotating the nodes into each other,



$$(6.150)$$

This symmetry interchanges the two spinor representations and the defining one, so this tells us they are all eight-dimensional.

6.11 Young Tableaux for $SU(n)$

We will now discuss a convenient way to represent irreducible representations of $\mathfrak{su}(n)$ by Young tableaux. The central point is that we can write all fundamental representations as antisymmetric tensor products of the defining representation. Since furthermore the fundamental representations are the building blocks of all representations (via tensor products), all representations are contained in some tensor products of the \mathfrak{n} .

We can identify the k -fold tensor product representation with the tensor components $\psi^{i_1 \dots i_k}$ as in

$$|\Psi\rangle = \psi^{i_1 \dots i_p} |i_1\rangle \otimes \dots \otimes |i_p\rangle, \quad (6.151)$$

where we have dropped the \mathfrak{n} from the kets. The algebra element X_i^j acts on the tensor components as

$$\psi^{i_1 \dots i_p} \longmapsto \psi^{j i_2 \dots i_p} X_j^{i_1} + \dots + \psi^{i_1 \dots i_{p-1} j} X_j^{i_p}. \quad (6.152)$$

(This comes about because we think of the kets $|i_1\rangle$ as lower-index objects which transform as $|i\rangle \mapsto X_i^j |j\rangle$, and then we can shift the action of X to the tensor components to obtain the expression.) Hence, in particular, symmetry or antisymmetry under exchange of indices is preserved, and thus a p -index tensor forms a representation of the symmetric group S_p . But we know all the irreducible representations of S_p – they are labelled by Young tableaux. Hence we find the important conclusion that the irreducible representations of $\mathfrak{su}(n)$ are determined by Young tableaux!

There is one slight difference with respect to the S_p case regarding the dimensions. This is most apparent for the totally symmetric combination with tableau $\square \square \dots \square \square$. This is always one-dimensional for S_p , because the permuted objects are distinct. Here, however, the indices can be equal, and so the totally symmetric representations have dimensions larger than one. For example, the totally symmetric three-index tensor of $\mathfrak{su}(2)$ has independent components ψ^{111} , ψ^{112} , ψ^{122} and ψ^{222} and is hence four-dimensional. This is reflected in a changed hook rule. Recall that for S_p , the dimension of a representation was $d = p!/H$, where H was the hook factor of the associated diagram. Here the hook factor is the same, but the numerator is now determined in the following way: Draw the diagram. In the upper left box, write an n (for representations of $\mathfrak{su}(n)$). Then increase the number in each box by one when you go right, and decrease it by one when going down. The numerator is then the product of all these numbers.

To illustrate this, consider again $\mathfrak{su}(3)$. The representations with one, two or three indices, and their dimensions are:

$$\begin{array}{l} \boxed{3} \quad d = \frac{3}{1} = 3, \\ \begin{array}{|c|} \hline \boxed{3} \\ \hline \boxed{2} \\ \hline \end{array} \quad d = \frac{6}{2} = 3, \quad \begin{array}{|c|c|} \hline \boxed{3} & \boxed{4} \\ \hline \end{array} \quad d = \frac{12}{2} = 6 \\ \begin{array}{|c|} \hline \boxed{3} \\ \hline \boxed{2} \\ \hline \boxed{1} \\ \hline \end{array} \quad d = \frac{6}{6} = 1, \quad \begin{array}{|c|c|} \hline \boxed{3} & \boxed{4} \\ \hline \boxed{2} \\ \hline \end{array} \quad d = \frac{24}{3} = 8, \quad \begin{array}{|c|c|c|} \hline \boxed{3} & \boxed{4} & \boxed{5} \\ \hline \end{array} \quad d = \frac{60}{6} = 10 \end{array} \quad (6.153)$$

We see that the $\bar{\mathbf{3}}$ is the two-index antisymmetric \square and the adjoint is the \square . This is a general feature: The $(n - 1)$ -fold antisymmetric representation is the conjugate of the defining representation, and the adjoint is the symmetric product of the defining and its conjugate. Furthermore, the n index antisymmetric representation is the trivial (singlet) one, as is obvious from symmetry. This also implies that you can erase any columns with n boxes from a Young tableau without changing the representation.

There is a simple way to go from the Dynkin labels to the Young tableau and back: The k -th Dynkin label is the number of columns of height k . Hence e.g. the Dynkin labels for the (non-singlet) three-index representations of $\mathfrak{su}(3)$ are $\boxed{1\ 1}$ and $\boxed{3\ 0}$.

Finally, note that to find the conjugate representation, one simply has to take a Dynkin diagram and complete it to a rectangle of height n . The object you have to add is the Dynkin diagram of the conjugate representation. For example, the conjugate of


(6.154)

in $\mathfrak{su}(5)$ is


(6.155)

because they form a rectangle together,


(6.156)

This automatically reproduces the conjugation rule for the Dynkin labels, $(q_1, \dots, q_{n-1}) = (q_{n-1}, \dots, q_n)$.

6.11.1 Products of Representations

Young tableaux also provide a nice way of determining the irreducible components of products of representations. For time reasons, here we just sketch the recipe:

1. Draw the Young tableaux of the two representations. In one of the tableaux, label the boxes in the first row by an a , in the second row by a b and so on.

This is to avoid overcounting and to weed out tableaux that are forbidden by the original symmetries: The product goes by stacking the boxes of one of the tableaux

onto the other one. In this way, certain product tableaux arise several times, and the labelling gives a method of kicking out the duplicates. Furthermore, one possibly obtains diagrams which violate the symmetries of the original tableaux, which are also eliminated.

Clearly, there are two choices of which diagram to label with the letters (unless both diagrams are equal), and usually one of them is more convenient. Often, but not always, this is the smaller one, because you have less boxes to move around.

2. Stack the a boxes onto the other diagram while keeping it a legal Young tableau, and without putting two a 's in the same column. In particular, there should be no columns with more than n boxes. Take the direct sum of all those diagrams.

Repeat with the b boxes on each of the diagrams thus obtained, and so on.

3. To kick out the forbidden and duplicate diagrams, read each diagram backwards, starting with the first line (i.e. begin with the first line backwards, the second line backwards, and so on). Delete all diagrams in which you encounter at some point in this procedure more b 's than a 's, more c 's than b 's etc.

Now you can remove all columns of n boxes, and the remaining diagrams are the irreducible components in the tensor product.

As an example consider the product $\mathbf{8} \otimes \bar{\mathbf{3}}$ in $\mathfrak{su}(3)$:

$$\begin{aligned}
 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline a & a \\ \hline b & \\ \hline \end{array} &= \left(\begin{array}{|c|c|} \hline & a \\ \hline & \\ \hline a & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline & & a \\ \hline & a & a \\ \hline & & \\ \hline \end{array} \right) \otimes \begin{array}{|c|} \hline \text{diagonal} \\ \hline b \\ \hline \end{array} \\
 &= \begin{array}{|c|c|} \hline & a \\ \hline & b \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline a & b \\ \hline & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline & & a \\ \hline & a & a \\ \hline & & b \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline a & a \\ \hline b & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & a \\ \hline & a \\ \hline b & \\ \hline \end{array} \\
 &= \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array}
 \end{aligned}$$

In other words, $\mathbf{8} \otimes \bar{\mathbf{3}} = \mathbf{15} \oplus \mathbf{6} \oplus \bar{\mathbf{3}}$. Note that the dimensions match. Here this order is simpler because at the first step there are only two possibilities.

6.12 Subalgebras

Often in physics one encounters the phenomenon of symmetry breaking – the symmetry of a system is reduced in some state. Group-theoretically, this means that the symmetry group G (or its algebra \mathfrak{g}) is reduced to some subgroup H (or subalgebra \mathfrak{h}). Then also the representations of G will in general not be irreducible under the action of H , but rather branch into several representations of H . Hence we will now analyse the possible subalgebras of the Lie algebras we considered, and how to find the branching rules.

Recall that a subalgebra \mathfrak{h} is a subspace of \mathfrak{g} which is closed under the Lie bracket. In the Cartan-Weyl basis, the algebra is generated by the Cartan generators H_i and the simple roots E_α , and there are $r = \text{rank } \mathfrak{g}$ of each. Now a simple way to generate a subalgebra is to simply discard one simple root $E_{\hat{\alpha}}$. The algebra still closes: The Cartan generators are still diagonal, and since $\hat{\alpha}$ is simple, it will never show up in the commutators of other roots. Of course, in this way one also loses a certain number of other (non-simple) roots, which contain $\hat{\alpha}$ in their simple root expansion. More importantly, the algebra is not simple anymore:

- We now have $r - 1$ simple roots in an r -dimensional space, so there will be a vector ξ which is orthogonal to all simple roots, and so $\xi \cdot \vec{H}$ is a generator which commutes with everything, so it generates a $U(1)$ factor.
- Even after removing $\xi \cdot \vec{H}$, the remaining algebra may be simple or semisimple, that is, it may decompose into disjoint simple pieces.

Hence, this procedure generally breaks $G \rightarrow H \times U(1)$, where the rank of H is one lower than the rank of G . This is particularly easy to see in the Dynkin diagram: Just erase one node from the diagram of G , and what remains is the diagram of H . The Abelian ideal is not visible in the Dynkin diagram, so one has to keep that in mind.

As an example, consider $\mathfrak{su}(5)$. The diagram is $\bigcirc - \bigcirc - \bigcirc - \bigcirc$. Removing one of the end roots breaks $SU(5) \rightarrow SU(4) \times U(1)$, removing one of the middle ones breaks $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$, which is how one likes to break the unified gauge group in grand unified theories. One can explicitly embed this subalgebra in the generators of $\mathfrak{su}(5)$. Recall that $\mathfrak{su}(n)$ generators are traceless Hermitian $n \times n$ matrices. Then we can find three commuting subalgebras in $\mathfrak{su}(5)$, generated by

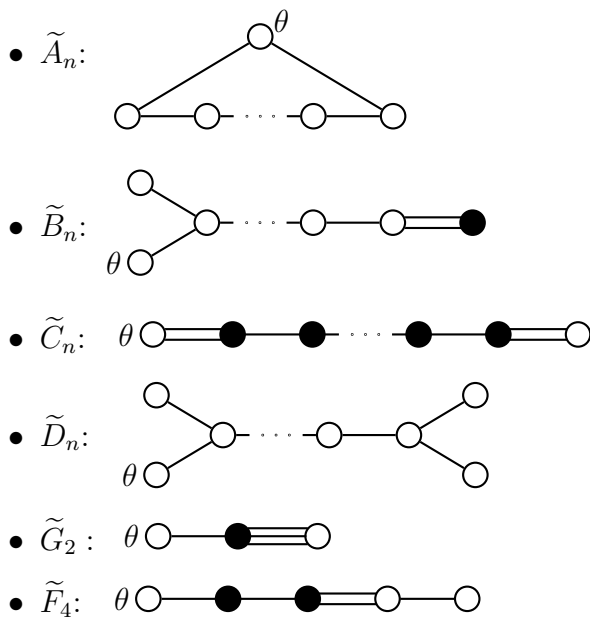
$$\begin{pmatrix} U_{\mathfrak{su}(3)} & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & U_{\mathfrak{su}(2)} \end{pmatrix}, \quad \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}, \quad (6.157)$$

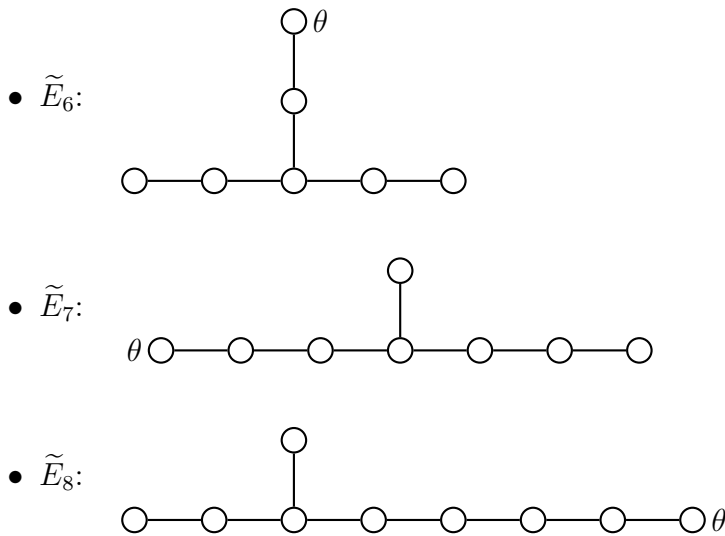
which correspond to $\mathfrak{su}(3)$, $\mathfrak{su}(2)$ and $U(1)$, respectively. Now it is obvious how these subalgebras act on the defining representation, which is a five-component vector: The first three components form the defining representation of $\mathfrak{su}(3)$ and the last two the defining of $\mathfrak{su}(2)$. Under the $U(1)$, they have charge 2 and -3 , respectively. (We have not discussed representations of $U(1)$ much, but they are simple – the group is Abelian, so all irreducible representations are one-dimensional, and they are determined by a “charge” q , which leads to a transformation $\psi \rightarrow e^{iq\alpha}\psi$ under the $U(1)$ element $e^{i\alpha}$. Since one can rescale α , the absolute value of the charges is not unique, but their ratio is.) This is summarised the so-called *branching rule*

$$\begin{aligned} SU(5) &\longrightarrow SU(3) \times SU(2) \times U(1) \\ \mathbf{5} &\longrightarrow (\mathbf{3}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{2})_{-3} . \end{aligned} \quad (6.158)$$

The convention is that $U(1)$ charges are given as subscripts, while the non-Abelian representations are denoted by their dimension. (Note that this is not necessarily unique, because representations of different groups can have the same dimension. For example, $SU(2)$, $SU(3)$, $SU(4)$ and $SU(5)$ all have a $\mathbf{10}$, so one has to keep in mind which group one is talking about.) Since this is the representation all others are built out of, this in principle fixes all branching rules of $SU(5)$ representations. In practice, one can look up these rules in the review by Slansky.

Now one can straightforwardly analyse all Dynkin diagrams and find out which subalgebras can be thus obtained. However, the semisimple piece of the resulting group always is of lower rank. Furthermore, while this procedure is simple and intuitive, it is not obvious that we find all subalgebras in this way. We will now discuss a more general approach. First, we define a *maximal subalgebra*: \mathfrak{h} is maximal if there is no (proper) subalgebra \mathfrak{h}_0 such that $\mathfrak{g} \supset \mathfrak{h}_0 \supset \mathfrak{h}$. Hence any non-maximal subalgebra of \mathfrak{g} is contained in some maximal one, so it suffices to study those. There are two kinds of maximal subalgebras, which are imaginatively called *regular* and *special*: The roots of regular subalgebras are a subset of the roots of the original algebra, while this is not true for special subalgebras. The special ones have to be looked for case by case, while the semisimple regular ones can be found in a systematic manner, which we will now do. The trick is to draw the *extended Dynkin diagram*, which is obtained by adding the most negative root θ , i.e. the root which is smaller than all other roots, to the set of simple roots. The most negative root has the property that $\theta - \alpha^i$ is not a root (for simple roots α^i), so the extended root system still satisfies all properties of a Π system except for linear independence. Hence, once we erase one root from the extended diagram, we again have a fully functional Π system determining a Lie algebra which is a maximal regular subalgebra of the original one (except in five cases with F_4 , E_7 and E_8). One can explicitly compute the most negative root and draw the extended Dynkin diagrams, which are:





You can find all possible regular subalgebras by deleting a node from the extended Dynkin diagram. In particular, you see that A_n does not have semisimple maximal subalgebras – deleting any node gives back A_n . Thus the true subalgebras of A_n contain a central element which generates a $U(1)$ subgroup – this fits with the fact that A_n is the lowest-dimensional simple Lie algebra of rank n . The other diagrams can be classified accordingly. Furthermore, there are five exceptions, where the subalgebra obtained by deleting a simple root is not maximal: Deleting the third simple root of F_4 , the third simple root of E_7 and the second, third or fifth simple root of E_8 .

It also seems odd that \tilde{D}_n does not contain a double line, so you cannot reach B_{n-1} as a subalgebra, while obviously $SO(2k-1) \subset SO(2k)$. However, B_{n-1} is always a special subalgebra of D_n , which is not found by this procedure.

6.13 Spinors and Clifford Algebra

We will now briefly mention a procedure to construct the strange spinorial representations of B_n and D_n . We already saw that they cannot be obtained as tensor products of the defining representations. However, there is an explicit way to find the generators in this representations via a so-called *Clifford algebra*.

To do so, first recall that the elements of $\mathfrak{so}(n)$ are the imaginary antisymmetric matrices. Hence as a basis one can choose matrices M_{ij} which have an i at column i , row j , and $-i$ at column j , row i , and zeroes otherwise. Then we clearly have $M_{ij} = -M_{ji}$, and the commutator of these generators then is

$$[M_{ij}, M_{kl}] = -i(\delta_{ik}M_{jl} - \delta_{il}M_{jk} - \delta_{jk}M_{il} + \delta_{jl}M_{ik}) . \quad (6.159)$$

So this is the Lie algebra we want to find a different representation of. To construct this, we introduce a new type of algebra, the Clifford algebra. It is generated by n operators Γ_i (and the unit matrix). These operators obey a “normal” associative product, so we

can write down terms like $\Gamma_i\Gamma_j$ (i.e. you can think of them as matrices), but the defining property is phrased in terms of the *anticommutator*:

$$\{\Gamma_i, \Gamma_j\} = \Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2\delta_{ij}\mathbb{1}. \quad (6.160)$$

You'll recognise this as the Euclidean version of the γ^μ matrices you use in particle physics, where on the right-hand side the Kronecker δ is replaced by the Minkowski metric $\eta_{\mu\nu}$. This is of course no coincidence, as the γ^μ matrices come in precisely when dealing with spinors (i.e. fermion fields).

Once you have a set of such Γ_i on your hands, you can immediately form generators of the $\mathfrak{so}(n)$ algebra by the commutator,

$$\Sigma_{ij} = \frac{i}{4} [\Gamma_i, \Gamma_j]. \quad (6.161)$$

You can check straightforwardly (though somewhat tediously) that the Σ_{ij} indeed satisfy the algebra (6.160). Since one can always find a representation of the Γ 's of dimension $2^{n/2}$, the Σ_{ij} provide a representation of the $\mathfrak{so}(n)$ algebra of that dimension. However, this is not necessarily irreducible: In even dimensions, i.e. $n = 2m$, one can form the operator

$$\Gamma_* = \Gamma_1 \cdots \Gamma_n \quad (6.162)$$

which anticommutes with all the Γ_i . Hence it is not proportional to the unit matrix, but still commutes with all the Σ_{ij} , so the representation is reducible. In fact, we can form projectors $P_\pm = \frac{1}{2}(! \pm \Gamma_*)$ which project onto the irreducible subspaces (of positive or negative *chirality*), which are irreducible as complex representations. In odd dimensions we can still consider Γ_* , but it is not so interesting: It commutes with all the Γ_i , and hence is already proportional to the unit matrix (by a variant of Schur's Lemma). Hence it cannot be used to form nontrivial projectors. (Again, this should be familiar to you from particle physics – there Γ_* is usually called γ^5 , and the corresponding subspaces are the left- and right-handed spinors.) Furthermore, for some dimensions the spinor representations are real (this is what is called a Majorana spinor). However, this reality constraint cannot always be imposed together with the chirality constraint. A detailed analysis of the possible spinor types is again straightforward, but tedious and we will not go into more detail in this lecture.

6.14 Casimir Operators

As a remark, we note that in a given representation, there is a special set of operators which commutes with all the generators of the algebra. If the algebra is semisimple, these operators are not elements of the Lie algebra, rather they are constructed from products of the generators. (Hence they are defined only in a representation, where the product of generators is well-defined.) In fancy parlance, they are elements of the *universal enveloping algebra*, which simply is the algebra generated by all generators

under the normal matrix product, or as formal power series in the abstract case. You know one of those operators for $\mathfrak{su}(2)$ – the total angular momentum operator

$$J^2 = J_1^2 + J_2^2 + J_3^2. \quad (6.163)$$

For a semisimple algebra of rank r , there are r independent such operators. One of them is always the generalisation of J^2 : Consider the Killing form g_{ij} . Since it is nondegenerate, we can find the inverse g^{ij} which is fixed by the requirement that $g_{ij}g^{jk} = \delta_i^k$. With the metric g_{ij} and its inverse we can now freely raise and lower indices. Then we define the operator

$$X = g^{ij}T_iT_j \quad (6.164)$$

and show that it commutes with all the generators:

$$\begin{aligned} [X, T_k] &= g^{ij} [T_iT_j, T_k] = g^{ij} (T_i [T_j, T_k] + [T_i, T_k] T_j) \\ &= ig^{ji} f_{jk}^l T_l T_i + ig^{ji} f_{ik}^l T_l T_i = ig^{ij} f_{jk}^l (T_i T_l + T_l T_i) \\ &= if_{jkl} (T^j T^l + T^l T^j) = 0. \end{aligned} \quad (6.165)$$

The expression vanishes because the lower-index structure constants are completely antisymmetric in their indices (cf. Eq. (6.9)) and the expression in brackets is symmetric. Since this Casimir operator exists for all Lie algebras, it is often called *the* Casimir operator.

The point of Casimir operators is that they commute with the whole algebra, and hence with the generated group. Hence, they are constant on any irreducible representation D , and their eigenvalue $c(D)$ can be used to distinguish representations. The actual value of $c(D)$ depends on the normalisation of the Casimir and the generators, but one can convert this into a canonical form, the so-called second-order index of the representation D , by suitably normalising:

$$I_D = \frac{\dim(D)}{\dim(\mathfrak{g})} c(D) \quad (6.166)$$

This is a quantity often encountered in loop calculations in QFT, and e.g. in the β functions that determine the running of the gauge couplings.

The other Casimir operators are not found as easily. In particular, they are polynomials of higher order in the generators (and can be chosen homogeneous). For the allowed Lie algebras they have the following orders:

$$\begin{array}{ll} A_n & 2, \dots, n+1 \\ B_n, C_n & 2, 4, \dots, 2n \\ D_n & 2, 4, \dots, 2n-2, n \\ G_2 & 2, 6 \\ F_4 & 2, 6, 8, 12 \\ E_6 & 2, 5, 6, 8, 9, 12 \\ E_7 & 2, 6, 8, 10, 12, 14, 18 \\ E_8 & 2, 8, 12, 14, 18, 20, 24, 30 \end{array} \quad (6.167)$$

For the A_n series, there is a easy way to find the third-order Casimir: It is given by

$$X_3 = d^{abc}T_aT_bT_c, \quad (6.168)$$

where

$$d_{abc} = \text{tr}(\{T_a, T_b\}T_c). \quad (6.169)$$

This is totally symmetric in its indices. The tensor d_{abc} is a invariant tensor, i.e. it does not change under the action of the Lie algebra, and it also has certain importance in particle physics (in four dimensions), because it measures the so-called anomalies, which need to vanish.

6.15 Summary

- Lie algebras always act on themselves by commutation – this is the adjoint representation.
- There is a special basis for the (complexified) Lie algebra, the Cartan–Weyl basis, in which the algebra is split into the Cartan generators and the roots. This is a generalisation of the $\mathfrak{su}(2)$ algebra, such that the Cartan generators are the analogues of J_3 in that their eigenvalues label the states, and the roots correspond to the raising and lowering operators.
- In a general representation, the eigenvalues of the Cartan generators are called weights. Every representation is uniquely determined by its highest weight, and can be explicitly constructed by the highest weight procedure.
- The roots are the weights in the adjoint representation. There is a special set of roots, the simple roots, from which the whole algebra can be reconstructed.
- The geometry of the simple roots is very constrained, in particular by the master formula. This allows for a complete classification of all possible simple Lie algebras: There are four infinite families and five exceptional cases.
- The infinite families correspond to the classical groups $SU(n)$, $SO(n)$ and $Sp(2n)$.